Algebraic Graph Theory

by

Paul Terwilliger

terwilli@math.wisc.edu

MEMO

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This is a lecture note based on a series of lectures given by Paul Terwilliger in 1993. The original is a hand written manuscript written by Paul Terwilliger.
This note was rewritten by Hiroshi Suzuki when he studied the lecture note during the following period.

Jan 13, 1995 — March 4, 1995.

He had chance to meet the author for a week right after he had read through the lecture note. The author clarified almost every thing he asked about. So even the part where he put "?", there seems to be no mathematical gap. But sometimes requires lengthy calculation.
In the last part each result has two numbers because the original lecture note has duplications.
This lecture note is already two years old so some parts are improved essentially, he suppose.

Hiroshi SUZUKI
hsuzuki@icu.ac.jp

THE SUBCONSITUENT ALGEBRA OF A GRAPH, THE THIN CONDITION, AND THE Q-POLYNOMIAL PROPERTY

0. Preface

This book attempts to prepare the way for an eventual classification of the graphs that are both thin and Q-polynomial. These graphs are distance-regular or bi-distance-regular, and since the distance-regular case is somewhat easier to handle, the focus will be on that case. (It is assumed the bi-distance-regular case is not too different). In the core of this book, we take a vertex x in a distance-regular graph, and study the irreducible modules for the subconstituent algebra $T(x)$ that have endpoint at most 2. (The modules with endpoint at least 3 seem too complicated to consider, and do not seem to play much of a role anyway). The thin condition and the Q-polynomial property each affect the structure of these modules, so these assumptions are first considered separately, and then jointly.

4d. Formula for $a_i(W)$, $x_i(W)$, $\gamma_i(W)$
4e. Feasibility conditions arising from the above constants being algebraic integers
4f. Feasibility conditions arising from $|a_i(W)| \leq a_{i+1}$ (?)
4g. A combinatorial characterization of the distance-regular graphs where every irreducible $T(x)$-module with endpoint 1 is thin.

5. Distance-regular graphs where each irreducible $T(x)$-module with endpoint 1 is thin

5a. Formulae for the multiplicities of the isomorphism classes of $T(x)$-modules with endpoint 1.
5b. The b_i's are determined by the c_i's and the structure of the 1st subconstituent.
5c. $a_1 = 0$ implies $a_i = 0\,(1 \leq i \leq D-1)$.
5d. Distance-regular graphs where the 1st subconstituent is strongly regular: restrictions on the parameters and possible classification (?)
5e. Distance-regular graphs where the 1st subconstituent has 4 distinct eigenvalues: restrictions on the parameters (?).
5f. Distance-regular graphs where the 1st subconstituent has 5 distinct eigenvalues: restrictions on the parameters (?).
5g. What minimal assumption (weaker than Q) implies Z (?)

6. The structure of a thin, irreducible $T(x)$-module with endpoint 2 in a distance-regular graph

6a. Similar to 4 (?)

7. The distance-regular graphs where each irreducible $T(x)$-module with endpoint at most 2 is thin

7a. The intersection numbers are determined by the structure of the 1st and 2nd subconstituents.
7b. The bipartite case.
7c. Classification of the examples where there are sufficiently few isomorphism classes of irreducible $T(x)$-modules with endpoint 1 or 2. (?)
7d. Classification of the almost-triply-regular graphs.

8. The Q-polynomial property

8a. Graphs that are Q-polynomial with respect to each vertex.(?)

9. Commutative association schemes

9a. The Bose-Mesner algebra M and the dual Bose-Mesner algebra M^*.
9b. The Krein parameters.
9c. The fundamental relations between M, M^*.
9d. An algebraic characterization of the Q-polynomial schemes.

9e. The representations of a commutative association scheme.
9f. A representation-theoretic characterization of the P- and Q-polynomial schemes.

10. Quantum Lie algebras

10a. The generators A, A^* satisfy two cubic polynomial equations.
10b. How these equations simplify in the thin case.
10c. Complete classification in the thin case.

11. Q-polynomial distance-regular graphs
11a. Formulae for the intersection numbers.
11b. A combinatorial characterization of the Q-polynomial distance-regular graphs that involves R, L, F.
11c. Formulae for the z_i constants.

12. Q-polynomial distance-regular graphs, cont.: The structure of an arbitrary irreducible $T(x)$-module with endpoint 1

12a. $E_1^*TE_1^*$ is commutative and has essentially one generator
12b. Description of the irreducible $T(x)$-modules with endpoint 1.
12c. There are at most 4 mutually non-isomorphic thin, irreducible $T(x)$-modules with endpoint 1.

13. The Q-polynomial distance-regular graphs of thin type: The ideal $T(x)E_1^*$

13a. The constant $\psi = \psi(x, y)$ is independent of the edge xy.
13b. $E_1^*TE_1^*$ is spanned by the all 1's matrix and 4 generalized adjacency matrices.
13c. $T(x)y = T(y)x$ if $\partial(x, y) = 1$. Complete desciption of this $T(x, y)$-module in terms of ψ and the intersection numbers.(?)
13d. The z_i are constant functions.
13e. Feasibility conditions forced by the integrality and non-negativity of the z_i.(?)
13f. Feasibility conditions forced by the integrality and non-negativity of the multiplicities of the irreducible $T(x)$-modules with endpoint 1. (?)

14. The Q-polynomial distance-regular graphs, cont.: The structure of an arbitrary irreducible $T(x)$-module with endpoint 2

14a. similar to 12(?)

15. The Q-polynomial distance-regular graphs of thin type: the ideal $T(x)E_2^*$

15a. Similar to 13(?)

16. The classification of the thin Q-polynomial distance-regular graphs with diameter at least (?)

17. Bi-distance-regular graphs

17a. If a bipartite graph is thin then so are the halved graphs

17b. For any thin $T(x)$-module W, $m_W(\theta) = m_W(-\theta)$.

17c. mimic the above sections 4-16(?) (I desperately hope that Q-polynomial bi-distance-regular graphs that are not already distance-regular do not exist)

Lecture 1 Wed. : Jan 20, 1993

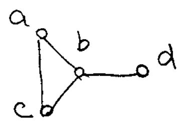
A *graph* (undirected, without loops or multiple edges) is a pair $\Gamma = (X, E)$, where

X = finite set (of vertices)

E = set of (distinct) 2-element subsets of X (= edges of Γ)

vertices $x, y \in X$ are *adjacent* $\Leftrightarrow$ $xy \in E$

Ex Γ :



$X = \{a, b, c, d\}$ $E = \{ab, ac, bc, bd\}$

Set $n = |X|$
$= \text{order of } \Gamma$

Pick field K ($= \mathbb{R}$ or $\mathbb{C}$)

$\mathrm{Mat}_X(K) \equiv$ K-algebra of all $n \times n$ matrices with entries in K.

(rows and columns are indexed by X)

Adjacency matrix $A \in \mathrm{Mat}_X(K)$

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E \\ 0 & \text{else} \end{cases}$$

Ex

$$A = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$M \equiv$ subalgebra of $\mathrm{Mat}_X(K)$ generated by A.
$\equiv$ Bose Mesner algebra of Γ

Set $V = K^n$ (the set of n-dim. column vectors)
(coordinates indexed by X)

Let $\langle , \rangle$ denote Hermitean inner product.

$$\langle u, v \rangle = \bar{u}^t \cdot v \qquad (u, v \in V)$$

V, $\langle , \rangle$ is the standard module of Γ

M acts on V: For $\forall x \in X$, write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow x\text{-position}$$

Then

$$A\hat{x} = \sum_{\substack{y \in X \\ xy \in E}} \hat{y}$$

Since A is a real symmetric matrix

$$V = V_0 + V_1 + \cdots + V_r \qquad \text{some } r \in \mathbb{Z}^{\geq 0}$$

(orthogonal direct sum of maximal A-eigenspaces).

Let $E_i \in \mathrm{Mat}_X(K)$ denote the orthogonal projection.

$$E_i : V \to V_i$$

Then $E_0, \cdots, E_r$ are the primitive idempotents of M.

$$M = \mathrm{Span}_K(E_0, \cdots, E_r)$$

$$E_i E_j = \delta_{ij} E_i \qquad \forall i, j.$$

$$E_0 + \cdots + E_r = I$$

Let θ_i denote eigenvalue of A for V_i $(\in \mathbb{R})$

WLOG.

$$\theta_0 > \theta_1 > \cdots > \theta_r$$

Let m_i = the multiplicity of θ_i

$= \dim V_i$

$= \operatorname{rank} E_i$.

Set $\mathrm{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \cdots, & \theta_r \\ m_0, & m_1, & \cdots, & m_r \end{pmatrix}$

<u>Problem</u> what can we say about Γ when $\mathrm{Spec}(\Gamma)$ is given?

The following Lemma 1 is an example of Problem.

$\forall x \in X$

$k(x) \equiv$ valency of x

$\equiv$ degree of x

$\equiv |\{y \mid y \in X, xy \in E\}|$

DEF. Γ: regular of valency $k \iff k(x) = k$ for $\forall x \in X$

LEMMA 1. With above notation,

(i) $\theta_0 \le \max\{k(x) \mid x \in X\} =: k^{max}$

(ii) Γ: regular of valency $k \Rightarrow \theta_0 = k$

PROOF. (i) WLOG $\theta_0 > 0$, else done.

Let $v := \sum_{x \in X} \alpha_x \hat{x}$ denote the eigenvector for θ_0.

Pick $x \in X$ with $|\alpha_x|$ maximal.

Then $|\alpha_x| \ne 0$.

Since $Av = \theta_0 v$,

$$\theta_0 \alpha_x = \sum_{\substack{y \in X \\ xy \in E}} \alpha_y .$$

So

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x|$$

$$\le \sum_{\substack{y \in X \\ xy \in E}} |\alpha_y|$$

$$\le k(x) |\alpha_x|$$

$$\le k^{max} |\alpha_x| .$$

(ii) All 1's vector $v = \sum_{x \in X} \hat{x}$ satisfies $Av = kv$.

Dual Bose Mesner algebra and Subconstituent algebra:

Let $x, y \in X$ and $\ell \in \mathbb{Z}^{\geq 0}$.

DEF. A <u>path</u> of length ℓ connecting x, y is a sequence
$x = x_0, x_1, \cdots, x_\ell = y$ $\quad x_i \in X \quad 0 \leq i \leq \ell$.

such that $\quad x_i x_{i+1} \in E \quad (0 \leq i \leq \ell - 1)$.

DEF. The <u>distance</u> $\partial(x, y)$ is the length of a shortest path connecting x and y.
$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$

DEF Γ : <u>connected</u> $\Leftrightarrow$ $\partial(x, y) < \infty \quad \forall x, y \in X$.

✻ From now on, <u>assume that Γ is connected</u> with $|X| \geq 2$.

Set $\quad d = \max\{\partial(x, y) \mid x, y \in X\}$
$\equiv$ the <u>diameter</u> of Γ.

Fix 'base' vertex $x \in X$.

DEF. $d(x)$ = diameter with respect to x.
$= \max\{\partial(x, y) \mid y \in X\}$
$\leq d$.

Observe

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^*$$

(orthogonal direct sum),

where

$$V_i^* = \mathrm{Span}_K(\hat{y} \mid \partial(x,y) = i)$$

$$\equiv V_i^*(x)$$

$\equiv$ <u>i-th</u> <u>subconstituent</u> <u>with</u> <u>respect</u> <u>to</u> <u>x</u>.

Let

$E_i^* = E_i^*(x)$ denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x)$$

View

$$E_i^*(x) \in \mathrm{Mat}_X(K).$$

So

$E_i^*(x)$ is diagonal, with yy entry

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{else} \end{cases} \qquad (y \in X)$$

Set

$$M^* \equiv M^*(x)$$

$$\equiv \mathrm{Span}_K(E_0^*(x), \cdots, E_{d(x)}^*(x))$$

(a commutative subalgebra of $\mathrm{Mat}_X(K)$)

$=$ <u>dual</u> <u>Bose</u> <u>Mesner</u> <u>algebra</u> w.r.t. x

DEF. $\Gamma = (X, E)$, x, M, $M^*(x)$ as above.
Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(K)$ generated by M, $M^*(x)$.

T is the <u>subconstituent</u> <u>algebra</u> of Γ w.r.t. x.

DEF. A <u>T-module</u> is any subspace $W \subset V$ s.t. $aw \in W$ for $\forall a \in T$ $\forall w \in W$.

<u>T-module</u> W is <u>irreducible</u>

$\Leftrightarrow$ $W \neq 0$, W does not properly contain a nonzero T-module.

For $\forall a \in \mathrm{Mat}_X(K)$, let a^* denote conjugate transpose.

<u>Observe</u> $\langle au, v \rangle = \langle u, a^* v \rangle$ $\forall a \in \mathrm{Mat}_X(K)$, $\forall u, v \in V$.

LEMMA 2. $\Gamma = (X, E)$, $x \in X$, $T \equiv T(x)$ as above.

(i) $a \in T \rightarrow a^* \in T$.

(ii) For any T-module $W \subseteq V$,
$W^\perp := \{ v \in V \mid \langle w, v \rangle = 0 \ \ \forall w \in W \}$ is a T-module.

(iii) V decomposes as an orthogonal direct sum of irreducible T-modules.

Proof.

(i) It is because T is generated by symmetric real matrices
$$A, \ E_0^*(x), \ E_1^*(x), \cdots, E_{d(x)}^*(x)$$

(ii) Pick $v \in W^\perp$ and $a \in T$,
it suffices to show that $av \in W^\perp$.

For $\forall w \in W$, $\langle w, av \rangle = \langle a^* w, v \rangle = 0$ as $a^* \in T$.

(iii) This is proved by the induction on the dimension of T-modules.

If W is an irreducible T-module of V,
$V = W + W^{\perp}$ (orthogonal direct sum)

<u>Problem</u> What does the structure of the $T(x)$-module tell us about Γ?

Study those Γ whose modules take 'simple' form.
The Γ's involved are highly regular.

$\boxed{\text{HS}}$ T: semisimple $\overset{\text{CR 25.2}}{\Leftrightarrow}$ left regular representation of T is completely red.
$1 = e_1 + \cdots + e_t$ e_i : projection $V \to W_i$
e : projection $e : V \to W$: W irred
$\langle a\ b \rangle_T = tr(a^t \bar{b})$: nondegenerate on T

T: semisimple Artinien $\Leftrightarrow$ T: Artinien with $J(A) = 0$
$\Leftarrow$ T: Artinien with no nonzero nilp. element
$\Leftarrow$ $T \subset Mat_X(\mathbb{C})$ $\forall a \in T$: normal

Lecture 2. Fri. Jan. 22, 1993

Use Perron Frobenius theory of nonnegative matrices to get informations on eigenvalues of a graph.

For today $K = \mathbb{R}$

Let $n \in \mathbb{Z}^{>0}$. Pick a symmetric matrix $C \in Mat_n(\mathbb{R})$

DEF C : reducible $\Leftrightarrow$ $\exists$ bipartition $\{1, 2, \cdots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets.) s.t. $C_{ij} = 0$ $\forall i \in X^+$, $\forall j \in X^-$

$$C \sim \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

DEF. C : bipartite $\Leftrightarrow$ $\exists$ bipartition $\{1, 2, \cdots, n\} = X^+ \cup X^-$ s.t. $C_{ij} = 0$ $\forall i, j \in X^+$ $\forall i, j \in X^-$

$$C \sim \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)$$

Note 1. If C is bipartite,
$\forall \theta$: eigenvalue of C $\rightarrow$ $-\theta$: eigenvalue of C s.t $mult(\theta) = mult(-\theta)$

Indeed let $C = \left(\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array}\right)$.

$$\left(\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \theta\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \left(\begin{array}{c|c} 0 & A \\ \hline B & 0 \end{array}\right)\begin{pmatrix} x \\ -y \end{pmatrix} = -\theta\begin{pmatrix} x \\ -y \end{pmatrix}$$

$Ay = \theta x$ $\quad Bx = \theta y$

Note 2 If C is bipartite, C^2 is reducible.

Note 3. C : irreducible, C^2 : reducible
If $C_{ij} \geq 0 \ \forall i, j$, C is bipartite.
(Exercise)

$\boxed{\text{HS}}$. For Note 1
Even if C is not symmetric

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}$$

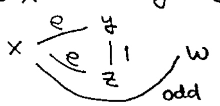
So the geometric multiplicities coincide.
How about the algebraic multiplicities ?

Note 3

Set $x \sim y \Leftrightarrow C_{xy} > 0$ May have loops.
$C^2_{xy} > 0 \Leftrightarrow \exists z$ s.t $x \sim z \sim y$

- C : irreducible $\Leftrightarrow$ $\Gamma(C)$: connected

$X^+ = \{ y \mid \exists \text{ path of even length from } x \text{ to } y \} \ni x$
$X^- = \{ y \mid \nexists \text{ path of even length from } x \text{ to } y \} \neq \emptyset$
$y, z \in X^+$ $y \sim z$ $w \in X^-$



※

So $e(X^+, X^+) = e(X^-, X^-) = 0$.

THEOREM 3 (Perron-Frobenius) Given $C \in \mathrm{Mat}_m(\mathbb{R})$ s.t.

(a) C: symmetric

(b) C: irreducible

(c) $C_{ij} \geq 0$ for all i, j

Let θ_0 be the maximal eigenvalue of C with eigenspace $V_0 \subseteq \mathbb{R}^n$

θ_r be the minimal eigenvalue of C with eigenspace $V_r \subseteq \mathbb{R}^n$

Then the following hold.

(i) Suppose $0 \neq v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$. Then $\alpha_i > 0$ for all i or $\alpha_i < 0$ for all i.

(ii) $\dim V_0 = 1$.

(iii) $\theta_r \geqslant -\theta_0$

(iv) $\theta_r = -\theta_0 \Leftrightarrow C$ is bipartite.

First we prove the following lemma.

LEMMA 4. Let $\langle \, , \rangle$ be the dot product in $V = \mathbb{R}^n$.

Pick a symmetric matrix $B \in \mathrm{Mat}_m(\mathbb{R})$.

Suppose all eigenvalues of B are nonnegative.

(i.e., B is positive semidefinite.)

Then there exist vectors $v_1, \cdots, v_n \in V$

such that $B_{ij} = \langle v_i, v_j \rangle$. $(1 \leq i, j \leq n)$

Proof. By elementary linear algebra, there exists an orthonormal basis $w_1, \cdots, w_n$ of V consisting of eigenvectors of B.

Say $B w_i = \theta_i w_i$.

Set the i-th column of P be w_i.

and $D = \mathrm{diag}(\theta_1, \cdots, \theta_n)$.

Then $P^t P = I$ and $BP = PD$.

Hence

$$B = PDP^{-1} = PDP^t = \Theta\Theta^t.$$

where

$$\Theta = P \cdot \mathrm{diag}(\sqrt{\theta_1}, \sqrt{\theta_2}, \cdots, \sqrt{\theta_n}) \in Mat_n(\mathbb{R})$$

Now let v_i be the i-th column of Θ^t.

Then

$$B_{ij} = v_i^t \cdot v_j = \langle v_i, v_j \rangle.$$

<u>Proof of THEOREM 3</u>

(i) Let $\langle\,,\,\rangle$ denote the dot product on $V = \mathbb{R}^n$.

Set

$$B = \theta_0 I - C$$

= symmetric matrix with eigenvalues $\theta_0 - \theta_i \geq 0$ (positive semidefinite)

$= (\langle v_i, v_j \rangle)_{1 \leq i, j \leq n}$

with some

$v_1, \cdots, v_n \in V$ by Lemma 4.

<u>Oberve</u> : $\sum_{i=1}^n \alpha_i v_i = 0$

(pf) $$\left\| \sum_{i=1}^n \alpha_i v_i \right\|^2 = \left\langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \right\rangle$$

$$= (\alpha_1, \cdots, \alpha_n) B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$= v^t B v$$

$$= 0$$

(Since $Bv = (\theta_0 I - C) v = 0$.)

Now set

$s = \#$ of indices i, where $\alpha_i > 0$..

Replacing v by $-v$ if necessary

WLOG. $s \geq 1$.

We want to show $s = n$

Assume $s < n$.

WLOG. $\alpha_i > 0 \qquad 1 \leq i \leq s$

$\qquad\quad \alpha_i \leq 0 \qquad s+1 \leq i \leq n$.

Set

$$\begin{aligned} p &= \alpha_1 v_1 + \cdots + \alpha_s v_s \\ &= -\alpha_{s+1} v_{s+1} - \cdots - \alpha_n v_n \end{aligned}$$

Then for $i = 1, \cdots, s$

$$\begin{aligned} \langle v_i, p \rangle &= \sum_{j=s+1}^{n} -\alpha_j \underbrace{\langle v_i, v_j \rangle}_{"B_{ij}} \\ &= \sum_{j=s+1}^{n} (-\alpha_j)(-c_{ij}) \qquad\qquad B = \theta_0 I - C \\ &\leq 0 \end{aligned}$$

Hence

$$\begin{aligned} 0 &\leq \langle p, p \rangle \\ &= \sum_{i=1}^{s} \underset{\substack{\vee \\ 0}}{\alpha_i} \underset{\substack{\wedge \\ 0}}{\langle v_i, p \rangle} \\ &\leq 0.. \end{aligned}$$

Thus we have

$$\langle p, p \rangle = 0 \quad \text{ad}$$
$$p = 0.$$

For $j = s+1, \cdots, n$

$$\begin{aligned} 0 &= \langle p, v_j \rangle \\ &= \sum_{i=1}^{s} \underset{\substack{\vee \\ 0}}{\alpha_i} \underbrace{\langle v_i\ v_j \rangle}_{} (-c_{ij}) \\ &\leq 0 \end{aligned}$$

Therefore

$$0 = \langle v_i, v_j \rangle = -C_{ij}$$

for $1 \le i \le s, \quad s+1 \le j \le n$

Since C is symmetric

$$C = \begin{array}{c|c} & \begin{array}{cc} 1 \cdots s & s{+}1 \cdots n \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ s \\ s{+}1 \\ \vdots \\ n \end{array} & \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right) \end{array}$$

Thus C is reducible, which is not the case.

Hence $s = n$

Proof of Thm 3 (ii)

Suppose $\dim V_0 \ge 2$. Then

$$\dim\left(V_0 \cap \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^{\perp} \right) \ge 1$$

So there is a vector

$$0 \ne v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$$

with $\alpha_1 = 0$. This contradicts (i).

Now pick

$$0 \ne w = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in V_r$$

Proof of Thm 3 (iii)

Suppose $\theta_r < -\theta_0$.

Since the eigenvalues of C^2 are the squares of those of C,

θ_r^2 is the maximal eigenvalue of C^2

Also we have

$$C^2 w = \theta_r^2 w.$$

Observe that C^2 is irreducible.
(otherwise C is bipartite by Note 3
and we must have $\theta_r = -\theta_0$)
Therefore. $\beta_i > 0$ for all i or
$\beta_i < 0$ for all i.
We have

$$\langle v, w \rangle = \sum_{i=1}^{n} \alpha_i \beta_i \neq 0$$

This is a contradiction as $V_0 \perp V_r$.

Proof of Thm 3 (iv)

$\Rightarrow$ Let $\theta_r = -\theta_0$.
Then $\theta := \theta_r^2 = \theta_0^2$ is the maximal
eigenvalue of C^2 and
v, w are linearly independent eigenvector
for θ for C^2
Hence for C^2,

$$\text{mult}(\theta) \geq 2$$

Thus by (ii), C^2 must be reducible.
Therefore C is bipartite by Note 3

$\Leftarrow$ This is Note 1

Let $\Gamma = (X, E)$ be any graph

DEF. Γ is bipartite $\Leftrightarrow$ The adjacency matrix A is bipartite

$\Leftrightarrow$ X can be written as a disjoint union of X^+ and X^- such that X^+, X^- contain no edges of Γ.

COR. 5. For any (connected) graph Γ with

$$Spec(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \cdots, & \theta_r \\ m_0, & m_1, & \cdots & m_r \end{pmatrix}$$

with $\theta_0 > \theta_1 > \cdots > \theta_r$.

Let V_i be the eigenspace of θ_i.

Then the following hold

(i) Suppose $0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0 \subseteq \mathbb{R}^n$

Then $\alpha_i > 0$ for all i or
$\alpha_i < 0$ for all i.

(ii) $m_0 = 1$

(iii) $\theta_r \geq -\theta_0$

(iv) $\theta_r = -\theta_0$ if and only if Γ is bipartite

In this case

$$\left\{ \begin{array}{l} -\theta_i = \theta_{r-i} \\ m_i = m_{r-i} \end{array} \right. \qquad (0 \leq i \leq r)$$

Proof This is a direct consequences of Theorem 3 and Note 1.

Lecture 3 Mon. Jan 25, 1993

Given graphs $\Gamma = (X, E)$, $\Gamma' = (X', E')$.

DEF. A map $\sigma : X \to X'$ is an isomorphism of graphs whenever

(i) σ is 1-1, onto

(ii) $xy \in E \leftrightarrow \sigma x\, \sigma y \in E'$ $\quad (\forall x, y \in X)$

<u>We do not distinguish between isomorphic graphs</u>

DEF. Suppose $\Gamma = \Gamma'$. Above isomorphism σ is called an <u>automorphism</u> of Γ.
The set $\mathrm{Aut}(\Gamma)$ of all automorphisms of Γ becomes a finite group under composition

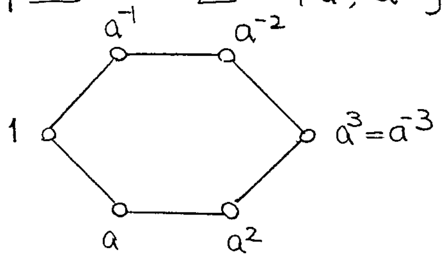
DEF. If $\mathrm{Aut}(\Gamma)$ acts transitive on X, Γ is called <u>vertex transitive</u>.

Example. The Cayley graphs:
G: any finite group.
Δ: any generating set for G
s.t. $1_G \notin \Delta$, and $g \in \Delta \to g^{-1} \in \Delta$

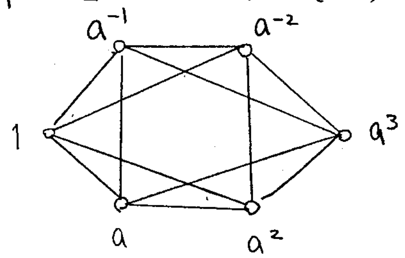
DEF. Cayley graph $\Gamma = \Gamma(G, \Delta)$:
$X = G$
$$E = \{ (h_1, h_2) \mid h_1, h_2 \in G,\ h_1^{-1} h_2 \in \Delta \}$$
$$= \{ (h, hg) \mid h \in G,\ g \in \Delta \}.$$

Let $G = \langle a \mid a^6 = 1 \rangle$

Example 1 $\Delta = \{a, a^{-1}\}$

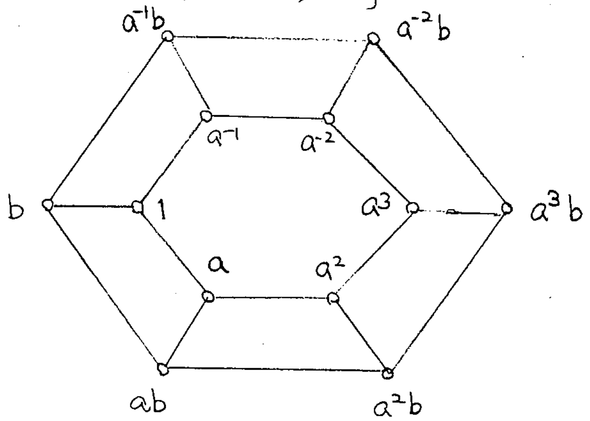


Example 2 $\Delta = \{a, a^{-1}, a^2, a^{-2}\}$



Example 3 $G = \langle a, b \mid a^6 = 1 = b^2,\ ab = ba \rangle$

$\Delta = \{a, a^{-1}, b\}$



$\boxed{\text{HS}}$ $\mathrm{Aut}\,\Gamma \simeq D_6 \times Z_2$

contains two regular subgroups isomorphic to D_6 and $Z_6 \times Z_2$ and Γ is obtained as Cayley graphs by two ways.

Cayley graphs are vertex transitive indeed.

Theorem 6 (i) For any Cayley graph $\Gamma = \Gamma(G, \Delta)$, the map

$$G \to \mathrm{Aut}(\Gamma)$$
$$g \to \hat{g}$$

is an injective homomorphism of groups, where

$$\hat{g}(x) = gx. \quad \forall g \in G, \ \forall x \in X \ (= G)$$

Also, the image $\hat{G}$ is <u>regular</u> on X, i.e., $\hat{G}$ acts transitively on X with trivial vertex stabilizers.

(ii) For any graph $\Gamma = (X, E)$. Suppose there exists a subgroup $G \leq \mathrm{Aut}(\Gamma)$ that is regular on X. Pick $x \in X$, and let

$$\Delta = \{ g \in G \mid (x, g(x)) \in E \}.$$

Then $1 \notin \Delta$, $g \in \Delta \to g^{-1} \in \Delta$ and Δ generates G.

Moreover, $\Gamma \cong \Gamma(G, \Delta)$

Proof. (i) Let $g \in G$.

We want to show that $\hat{g} \in \mathrm{Aut}(\Gamma)$

Let $\forall h_1, h_2 \in X = G$ Then

$$\begin{aligned}(h_1, h_2) \in E &\leftrightarrow h_1^{-1} h_2 \in \Delta \\ &\leftrightarrow (gh_1)^{-1}(gh_2) \in \Delta \\ &\leftrightarrow (gh_1, gh_2) \in E \\ &\leftrightarrow (\hat{g}(h_1), \hat{g}(h_2)) \in E.\end{aligned}$$

Hence $\hat{g} \in \mathrm{Aut}(\Gamma)$

<u>Observe</u> $g \to \hat{g}$ is a homomorphism of groups:

$$\hat{1}_G = 1 \qquad \widehat{g_1 g_2} = \hat{g}_1 \cdot \hat{g}_2.$$

Observe: $g \to \hat{g}$ is 1-1. :
If $\hat{g}_1 = \hat{g}_2$, then
$$g_1 = \hat{g}_1(1_G) = \hat{g}_2(1_G) = g_2$$

Observe: $\hat{G}$ is regular on X :
Clear by construction

(ii) · $1 \notin \Delta$;
Since Γ has no loops, $(x, 1 \cdot x) \notin E$

· $g \in \Delta \to g^{-1} \in \Delta$:
$g \in \Delta \to (x, g(x)) \in E$
$\to E \ni (g^{-1}(x), g^{-1}(g(x))) = (g^{-1}(x), x)$

· Δ generates G:
Suppose $\langle \Delta \rangle \subsetneq G$.
Let $\hat{X} = \{ g(x) \mid g \in \langle \Delta \rangle \} \subsetneq X$
($\hat{X} \subsetneq X$, as G acts regularly on X.)
Since Γ is connected, there exists $y \in \hat{X}$
and $z \in X \setminus \hat{X}$ with $yz \in E$.
Let $y = g(x)$ $\quad g \in \langle \Delta \rangle$
$\quad z = h(x)$ $\quad h \in G \setminus \langle \Delta \rangle$
Then $(y, z) = (g(x), h(x)) \in E$
$\to (x, g^{-1}h(x)) \in E$.
$\to g^{-1}h \in \langle \Delta \rangle$.
$\to h \in \langle \Delta \rangle$.
This is a contradiction
Therefore Δ generates G.

Let $\Gamma' = (X', E')$ denote $\Gamma(G, \Delta)$.

We shall show that

$$\begin{aligned} \theta : X' &\to X \\ g &\mapsto g(x) \end{aligned}$$

is an isomorphism of graphs.

- $\underline{\theta \text{ is } 1\text{-}1}$:

  $\forall h_1, h_2 \in X' = G$

$$\begin{aligned} \theta(h_1) = \theta(h_2) &\to h_1(x) = h_2(x) \\ &\to h_2^{-1} h_1(x) = x \\ &\to h_2^{-1} h_1 \in \mathrm{Stab}_G(x) = 1_G \\ &\to h_1 = h_2 \end{aligned}$$

  $(\mathrm{Stab}_G(x) = \{ g \in G \mid g(x) = x \})$

- $\underline{\theta \text{ is onto}}$:

  Since G is transitive

$$X = \{ g(x) \mid g \in G \} = \theta(X') = \theta(G).$$

- $\underline{\theta \text{ respects adjacency}}$.

  $\forall h_1, h_2 \in X' = G$.

$$\begin{aligned} (h_1, h_2) \in E' &\leftrightarrow h_1^{-1} h_2 \in \Delta \\ &\leftrightarrow (x, h_1^{-1} h_2(x)) \in E \\ &\leftrightarrow (h_1(x), h_2(x)) \in E \\ &\leftrightarrow (\theta(h_1), \theta(h_2)) \in E \end{aligned}$$

Therefore θ is an isomorphism between graphs $\Gamma(G, \Delta)$ and $\Gamma = (X, E)$

How to compute the eigenvalues of the Cayley graph of an abelian group.

Let G be any finite abelian group.
Let $\mathbb{C}^*$ be the multiplicative group on $\mathbb{C}-\{0\}$.

DEF A (linear) G-character is any group homomorphism $\theta : G \to \mathbb{C}^*$.

Example $G = \langle a \mid a^3 = 1 \rangle$ has 3 characters $\theta_0, \theta_1, \theta_2$

$\theta_i(a^j)$	1	a	a^2
θ_0	1	1	1
θ_1	1	ω	ω^2
θ_2	1	ω^2	ω

$$\left(\omega = \frac{-1+\sqrt{-3}}{2} \right)$$

Here ω is a primitive cube root of 1 in $\mathbb{C}^*$,
(i.e., $1+\omega+\omega^2=0$)

For arbitrary G, let $X(G)$ be the set of all characters of G.

Observe: For $\forall \theta_1, \theta_2 \in X(G)$, one can define product $\theta_1\theta_2$:

$$\theta_1\theta_2(g) = \theta_1(g)\,\theta_2(g) \qquad \forall g \in G.$$

Then $\theta_1\theta_2 \in X(G)$

Observe: $X(G)$ with this product is an (abelian) group.

LEMMA 7. The groups G and $X(G)$ are isomorphic for all finite abelian groups G.

Proof G is a direct sum of cyclic groups

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m,$$

where $G_i = \langle a_i \mid a_i^{d_i} = 1 \rangle$ $(1 \le i \le m)$

Pick any element ω_i of order d_i in $\mathbb{C}^*$ (i.e., a primitive d_i-th root of 1.)

Define

$$\theta_i : G \longrightarrow \mathbb{C}^*$$
$$a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_n} \longrightarrow \omega_i^{\varepsilon_i} \qquad (0 \le \varepsilon_i < d_i, \ 1 \le i \le m).$$

Then $\theta_i \in X(G)$. (Exercise).

Claim There exists an isomorphism of groups $G \to X(G)$ that sends $a_i \to \theta_i$.

Observe $\theta_i^{d_i} = 1$:

For $\forall g = a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \in G$,

$$\begin{aligned} \theta_i^{d_i}(g) &= (\theta_i(g))^{d_i} \\ &= (\omega_i^{\varepsilon_i})^{d_i} \\ &= (\omega_i^{d_i})^{\varepsilon_i} \\ &= 1 \end{aligned}$$

Observe If $\theta_1^{\varepsilon_1}\theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m} = 1$ $(0 \le \varepsilon_i < d_i, \ 1 \le i \le m)$, then $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = 0$.

(Pf.)

$$\begin{aligned} 1 &= \theta_1^{\varepsilon_1}\theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m}(a_i) \\ &= \omega_i^{\varepsilon_i} \end{aligned}$$

Since ω_i is a primitive d_i-th root of 1, $\varepsilon_i = 0$ for $1 \le i \le m$.

<u>Observe</u> $\theta_1, \cdots, \theta_m$ generate $X(G)$;

Pick $\theta \in X(G)$

Since $a_i^{d_i} = 1$,

$$1 = \theta(a_i^{d_i}) = \theta(a_i)^{d_i}.$$

Hence $\theta(a_i) = \omega_i^{\varepsilon_i}$ for some ε_i $(0 \le \varepsilon_i < d_i)$

Now $\theta = \theta_1^{\varepsilon_1} \cdots \theta_m^{\varepsilon_m}$,

since these are both equal to $\omega_i^{\varepsilon_i}$ at a_i $(1 \le i \le m)$

Therefore

$$G \to X(G)$$
$$a_i \to \theta_i$$

is an isomorphism of groups

Note: The correspondence above is clearly a group homomorphism

Lecture 4 Wed. Jan. 27, 1993

THEOREM 8 Given a Cayley graph $\Gamma = \Gamma(G, \Delta)$.
(View standard module

$$V \equiv \mathbb{C}G \quad (\text{the group algebra}), \quad \text{so}$$

$$\langle \sum_{g \in G} \alpha_g g, \sum_{g \in G} \beta_g g \rangle = \sum_{g \in G} \alpha_g \overline{\beta_g} \quad (\alpha_g, \beta_g \in \mathbb{C}) \quad)$$

For any $\theta \in X(G)$, write

$$\hat{\theta} = \sum_{g \in G} \theta(g^{-1}) g$$

then

(i) $$\langle \hat{\theta}_1, \hat{\theta}_2 \rangle = \begin{cases} |G| & \text{if } \theta_1 = \theta_2 \\ 0 & \text{if } \theta_1 \neq \theta_2. \end{cases} \qquad \theta_1, \theta_2 \in X(G).$$

In particular, $\{\hat{\theta} \mid \theta \in X(G)\}$ forms a basis for V.

(ii) $$A\hat{\theta} = \Delta_\theta \hat{\theta} \qquad (\theta \in X(G)),$$

where A is the adjacency matrix and

$$\Delta_\theta = \sum_{g \in \Delta} \theta(g)$$

In particular, the eigenvalues of Γ are precisely $\{\Delta_\theta \mid \theta \in X(G)\}$.

Proof (i)

<u>Claim</u> For $\forall \theta \in X(G)$:

$$s := \sum_{g \in G} \theta(g^{-1}) = \begin{cases} |G| & \text{if } \theta = 1 \\ 0 & \text{if } \theta \neq 1. \end{cases}$$

(pf.) Clear if $\theta = 1$.

Let $\theta \neq 1$. Then $\theta(h) \neq 1$ for some $h \in G$.

$$s\,\theta(h) = \Big(\sum_{g \in G} \theta(g^{-1})\Big)\theta(h)$$

$$= \sum_{g \in G} \theta(g^{-1}h) = \sum_{g' \in G} \theta(g'^{-1}) = s$$

Since $\theta(h) \neq 1$, $s = 0$.

<u>Claim</u> $\theta(g^{-1}) = \overline{\theta(g)}$ for $\forall \theta \in X(G)$, $\forall g \in G$.

Since $\theta(g) \in \mathbb{C}$ is a root of 1,

$$|\theta(g)|^2 = \theta(g)\overline{\theta(g)} = 1$$

On the other hand, since θ is a homomorphism.

$$\theta(g)\theta(g^{-1}) = \theta(1) = 1.$$

Hence $\theta(g^{-1}) = \overline{\theta(g)}$

Now

$$\begin{aligned}
\langle \hat{\theta}_1, \hat{\theta}_2 \rangle &= \sum_{g \in G} \theta_1(g^{-1})\overline{\theta_2(g^{-1})} \\
&= \sum_{g \in G} \theta_1(g^{-1})\theta_2(g) \\
&= \sum_{g \in G} \theta_1\theta_2^{-1}(g^{-1}) \\
&= \begin{cases} |G| & \text{if } \theta_1\theta_2^{-1} = 1 \\ 0 & \text{if } \theta_1\theta_2^{-1} \neq 1 \end{cases}
\end{aligned}$$

Since $|G| = |X(G)|$ by Lemma 7 and $\hat{\theta}_i$'s are orthogonal nonzero elements in V, they form a basis of V.

(ii). Let $\Delta = \{g_1, \cdots, g_r\}$.

Then

$$\begin{aligned}
A\hat{\theta} &= A\Big(\sum_{g \in G} \theta(g^{-1})g\Big) \\
&= \sum_{g \in G} \theta(g^{-1})(gg_1 + \cdots + gg_r) \qquad (\Gamma(g) = \{gg_1, \cdots, gg_r\}) \\
&= \sum_{i=1}^{r}\Big(\sum_{g \in G} \theta(g^{-1})(gg_i)\Big) \\
&= \sum_{i=1}^{r}\Big(\sum_{g \in G} \theta(g_i g_i^{-1} g^{-1})(gg_i)\Big)
\end{aligned}$$

$$= \sum_{i=1}^{r} \left(\sum_{g \in G} \theta(g_i)\, \theta((g g_i)^{-1})\, g g_i \right)$$

$$= \sum_{i=1}^{r} \theta(g_i) \sum_{h \in G} \theta(h^{-1})\, h$$

$$= \Delta_\theta \cdot \hat{\theta}$$

Since $\{\hat{\theta} \mid \theta \in X(G)\}$ forms a basis, the eigenvalues of Γ are precisely

$$\{ \Delta_\theta \mid \theta \in X(G) \}$$

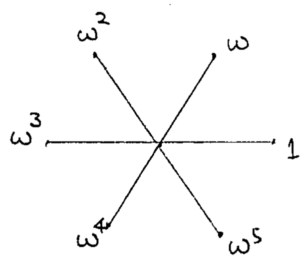
Example. $G = \langle a \mid a^6 = 1 \rangle$

$\Delta = \{a, a^{-1}\}$

Pick ω : primitive 6-th root of 1

$X(G) = \{ \theta^i \mid 0 \le i \le 5 \}$

$\theta(a) = \omega$, $\omega + \omega^{-1} = 1$



$\varphi \in X(G)$	$\varphi(a)$	$\Delta_\theta = \theta(a) + \theta(a)^{-1}$
1	1	2
θ	ω	$\omega + \omega^{-1} = 1$
θ^2	ω^2	-1
θ^3	$\omega^3 = -1$	-2
θ^4	ω^4	-1
θ^5	ω^5	1

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}$$

Example. D-cube $H(D,2)$

$$X = \{ (a_1, \cdots, a_D) \mid a_i \in \{1, -1\},\ 1 \le i \le D \}$$
$$E = \{ xy \mid x, y \in X,\ x, y : \text{different in exactly one coordinate} \}$$

Also $H(D,2)$ is a Cayley graph $\Gamma(G, \Delta)$, where,

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_D$$
$$G_i = \langle a_i \mid a_i^2 = 1 \rangle$$
$$\Delta = \{ a_1, \cdots, a_D \}$$

Home Work. : The spectrum of $H(D,2)$ is

$$\begin{pmatrix} \theta_0, \theta_1, \cdots & \theta_D \\ m_0, m_1, \cdots, & m_D \end{pmatrix}.$$

where,

$$\theta_i = D - 2i \qquad (0 \le i \le D)$$
$$m_i = \binom{D}{i}.$$

$\boxed{\text{HS}}$ $\theta \in X(G)$

$\theta : X \longrightarrow \{\pm 1\}$.

If $\nu(\theta) = |\{ i \mid \theta(a_i) = -1 \}|$,

$\Delta_\theta = D - 2\nu(\theta)$

Since there are $\binom{D}{\nu(\theta)}$ such θ,

we have the assertion.

We want to compute the subconstituent algebra for $H(D,2)$
First, we make a few observations about arbitrary graphs.

$\Gamma = (X, E)$ any graph
$A =$ adjacency matrix of Γ
$V \equiv$ standard module / $K = \mathbb{C}$

Fix a base $x \in X$.
Write $E_i^* \equiv E_i^*(x)$,
$T \equiv T(x)$
$=$ the algebra generated by
$A, E_0^*, E_1^*, \cdots$.

DEF. Let W be any irreducible T-module ($\subseteq V$)
The <u>endpoint</u> $r \equiv r(W)$ satisfies
$r = \min\{i \mid E_i^* W \neq 0\}$.
The <u>diameter</u> $d \equiv d(W)$ satisfies
$d = |\{i \mid E_i^* W \neq 0\}| - 1$

LEMMA 9. With the above notation, let W be an irreducible T-module. Then

(i) $E_i^* A E_j^* \begin{cases} = 0 & \text{if } |i-j|>1 \\ \neq 0 & \text{if } |i-j|=1 \end{cases}$ $\quad 0 \le i,j \le d(x)$

(ii) $A E_j^* W \subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W$
$0 \le j \le d(x)$. $\quad$ ($E_i^* W = 0$ if $i<0$ or $i>d(x)$)

(iii) $E_j^* W \begin{cases} \neq 0 & \text{if } r \le j \le r+d \\ = 0 & \text{if } 0 \le j < r \text{ or } r+d < j \le d(x) \end{cases}$

(iv) $E_i^* A E_j^* W \neq 0$ if $|i-j| = 1$ $\quad (r \le i,j \le r+d)$

Proof. (i) Pick $y \in X$ with $\partial(x,y) = j$.
We want to find $E_i^* A E_j^* \hat{y}$.

(Note. $E_j^* \hat{y} = \begin{cases} 0 & \text{if } \partial(x, y) \neq j \\ \hat{y} & \text{if } \partial(x, y) = j. \end{cases}$)

$$E_i^* A E_j^* \hat{y} = E_i^* A \hat{y}$$

$$= E_i^* \sum_{z \in X,\ yz \in E} \hat{z}$$

$$= \sum_{z \in X,\ yz \in E,\ \partial(x, y) = i} \hat{z} \qquad *$$

$$= 0 \quad \text{if } |i-j| > 1 \quad \text{by triangle inequality}$$

If $|i-j| = 1$, there exist $y, y' \in X$ such that
$\partial(x, y) = j$, $\partial(x, y') = i$, $yy' \in E$
by connectivity of Γ.
Hence $*$ contains $\hat{y}'$ and $* \neq 0$.

(ii) $$A E_j^* W = \left(\sum_{i=0}^{d(x)} E_i^* \right) A E_j^* W$$

$$= E_{j-1}^* A E_j^* W + E_j^* A E_j^* W + E_{j+1}^* A E_j^* W$$

$$\subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W$$

(iii) Suppose $E_j^* W = 0$ for some j $(r \leq j \leq r+d)$
Then $r < j$ by definition of r.
Set
$$\tilde{W} = E_r^* W + E_{r+1}^* W + \cdots + E_{j-1}^* W$$
Observe $0 \subsetneq \tilde{W} \subsetneq W$.
Also
$$A \tilde{W} \subseteq \tilde{W} \quad \text{by (ii)}$$
and
$$E_i^* \tilde{W} \subseteq \tilde{W} \quad \text{for } \forall i \quad \text{by construction}$$
Thus $T \tilde{W} \subseteq \tilde{W}$,
contradicting W being irreducible

Lecture 5 Fri. Jan 29, 1993

$\Gamma = (X, E)$: any graph

A : adjacency matrix

V : standard module $/ K = \mathbb{C}$

Fix a base $x \in X$. Write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$.

W: irreducible T-module with endpoint

$$r := \min\{ i \mid E_i^* W \neq 0 \}$$

diameter $\quad d := |\{ i \mid E_i^* W \neq 0 \}| - 1$

- We have $E_i^* W \neq 0 \quad r \le i \le r+d$

  $= 0 \quad 0 \le i < r$ or $r+d < i \le d(x)$.

Claim $\quad E_i^* A E_j^* W \neq 0$ if $|i-j| = 1$ $(r \le i, j \le r+d)$. (Lemma 9 (iv))

Suppose $E_{j+1}^* A E_j^* W = 0$ for some j $(r \le j < r+d)$.

Observe that

$$\tilde{W} = E_r^* W + \cdots + E_j^* W$$

is T-invariant with

$$0 \subsetneq \tilde{W} \subsetneq W.$$

Because $A\tilde{W} \subseteq \tilde{W}$ since $AE_j^* W \subseteq E_{j-1}^* W + E_j^* W$

$E_k^* \tilde{W} \subseteq \tilde{W} \quad \forall k$.

we have $T\tilde{W} \subseteq \tilde{W}$.

Similarly, suppose $E_{i-1}^* A E_i^* W = 0$ for some i $(r < i \le r+d)$

Set $\hat{W} = E_i^* W + \cdots + E_{r+d}^* W$ is a T-module.

with $0 \subsetneq \hat{W} \subsetneq W$.

DEF. Γ, E_i^*, T as above.
Irreducible T-modules W, W' are <u>isomorphic</u> whenever $\exists$ isomorphism $\sigma : W \to W'$ of vector spaces.
s.t. $a\sigma = \sigma a$ for $\forall a \in T$

Recall that the standard module V
$=$ orthogonal direct sum of irreducible T-modules
$W_1 + W_2 + \cdots$
given W in this list.
the <u>multiplicity</u> of W (in V) is
$|\{j \mid W_j \simeq W\}|$.

$\boxed{\text{HS}}$ It needs to mention the uniqueness of homog. decomp.

Now assume.
Γ : D-cube $H(D,2)$ $(D \geq 1)$
View $X = \{a_1 \cdots a_D \mid a_i \in \{1,-1\},\ 1 \leq i \leq D\}$
$E = \{xy \mid x, y \in X,\ x, y$ differ in exactly 1 coordinate.$\}$

Find T-modules.

<u>Observe</u> : $H(D,2)$ is bipartite $X = X^+ \cup X^-$ with
$X^+ = \{a_1 \cdots a_D \in X \mid \prod a_i > 0\}$
$X^- = \{a_1 \cdots a_D \in X \mid \prod a_i < 0\}$.

Observe : $\forall y, z \in X$
$\partial(y,z) = i \iff y, z$ differ in exactly i coordinates. $(0 \leq i \leq D)$
Hence the diameter of $H(D,2) = D$
$= d(x)$, $\forall x \in X$.

THEOREM 10. Let $\Gamma = H(D, 2)$ as above.
Fix $x \in X$, and write $E_i^* = E_i^*(x)$, $T = T(x)$.
Let W be an irreducible T-module with endpoint r, diameter d $(0 \le r \le r+d \le D)$.

(i). W has a basis $w_0, w_1, \cdots, w_d$
$(w_i \in E_{i+r}^* W \quad (0 \le i \le d))$
with respect to which the matrix representing A is

$$\begin{bmatrix} 0 & d & & & & \\ 1 & 0 & d-1 & & \text{\Large 0} & \\ & 2 & 0 & \ddots & & \\ & & 3 & \ddots & & \\ & & & \ddots & 2 & \\ & \text{\Large 0} & & d-1 & 0 & 1 \\ & & & & d & 0 \end{bmatrix}$$

(ii) $d = D - 2r$. In particular $0 \le r \le D/2$

(iii) Let W' denote an irreducible T-module with end point r'. Then W, W' are isomorphic as T-modules if and only if $r = r'$

(iv) The multiplicity of the irreducible T-module with end poind r is

$$\binom{D}{r} - \binom{D}{r-1} \qquad (\text{if } 1 \le r \le D/2)$$

and 1 if $r = 0$.

Proof Γ : is vertex transitive (Cayley graph)
Hence without loss of generality,
we may assume that
$$x = \underbrace{1\,1 \cdots\cdots 1}_{D}$$

Notation

Set $\Omega = \{1, 2, \cdots, D\}$

For $\forall S \subseteq \Omega$ Let

$\hat{S} = a_1 \cdots a_D \in X$ $\left(a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases} \right)$

In particular,

$\hat{\phi} = x$, and

$|S| = i \iff \partial(x, \hat{S}) = i \iff \hat{S} \in E_i^* V.$

DEF. $\forall S, T \subseteq \Omega$,

S covers $T \iff S \supseteq T$ and $|S| = |T| + 1$

Observe: $\hat{S}, \hat{T}$ are adjacent in Γ

$\iff T$ covers S or S covers T.

Define "raising matrix"

$$R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*.$$

Observe: $R E_i^* V \subseteq E_{i+1}^* V$ $\quad (0 \le i \le D) \quad (E_{D+1}^* V = 0)$

Indeed for $\forall S \subseteq \Omega$ with $|S| = i$

$$R\hat{S} = R E_i^* \hat{S} = E_{i+1}^* A \hat{S}$$

$$= \sum_{T_1 \subseteq \Omega,\ S \text{ covers } T_1} E_{i+1}^* \hat{T_1} + \sum_{T \subseteq \Omega\ T \text{ covers } S} E_{i+1}^* \hat{T}$$

$$= \sum_{T \subseteq \Omega\ T \text{ covers } S} \hat{T}.$$

Define "lowering matrix"

$$L = \sum_{i=0}^{D} E_{i-1}^* A E_i^*$$

Observe : $L E_i^* V \subseteq E_{i-1}^* V \qquad (0 \le i \le D) \qquad (E_{-1}^* V = 0)$.

Indeed for $\forall S \subseteq \Omega$

$$L\hat{S} = \sum_{\substack{T \subseteq \Omega \\ S \text{ covers } T}} \hat{T}$$

Observe $A = L + R$.

For convenience,

$$A^* = \sum_{i=0}^{D} (D-2i) E_i^*.$$

Claim (a) $LR - RL = A^*$
(b) $A^*L - LA^* = 2L$
(c) $A^*R - RA^* = -2R$

(In particular, Span (R, L, A^*) is a representation of Lie algebra $\mathfrak{sl}_2(\mathbb{C})$)

H.S

$\mathfrak{sl}_2(\mathbb{C}) = \{ X \in \mathrm{Mat}_2(\mathbb{C}) \mid \mathrm{tr}(X) = 0 \}$

$X, Y \in \mathfrak{sl}_2(\mathbb{C})$, $\quad [X, Y] = XY - YX$

$A^* \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $\quad L \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $\quad R \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Then these satisfy the relations (a) – (c) above

Proof of Claim

Apply both sides to $\hat{S}$ $(S \subseteq \Omega)$.
Say $|S| = i$.

(a) $(LR-RL)\hat{S} = L\Big(\sum_{\substack{T\subseteq\Omega \\ T \text{ covers } S}} \hat{T}\Big) - R\Big(\sum_{\substack{U\subseteq\Omega \\ S \text{ covers } U}} \hat{U}\Big)$

($D-i$ of them; i of them)

$$= (D-i)\hat{S} + \sum_{\substack{V\subseteq\Omega\ |V|=i \\ |S\cap V| = i-1}} \hat{V}$$

$$- \Big(i\hat{S} + \sum_{\substack{V\subseteq\Omega\ |V|=i \\ |S\cap V| = i-1}} \hat{V} \Big)$$

$$= (D-2i)\hat{S}$$

$$= A^*\hat{S}.$$

(b) $(A^*L - LA^*)\hat{S}$

$$= (D-2(i-1))L\hat{S} - (D-2i)L\hat{S} \qquad (\text{since } L\hat{S} \in E^*_{i-1}V)$$

$$= 2L\hat{S}.$$

(c) $(A^*R - RA^*)\hat{S}$

$$= (D-2(i+1))R\hat{S} - (D-2i)R\hat{S} \qquad (\text{since } R\hat{S} \in E^*_{i+1}V)$$

$$= -2R\hat{S}$$

Let W be an irreducible T-module with endpoint r and diameter d.
$(0 \le r \le r+d \le D)$

Proof of (i) and (ii)

Pick $0 \ne w \in E_r^* W$.

<u>Claim</u> $LRw = (D-2r)w$

(Pf.)
$$LRw = (A^* + RL)w \qquad \text{(by Claim (a))}$$
$$= A^* w \qquad (Lw \in E_{r-1}^* W = 0)$$
$$= (D-2r)w$$

Define
$$w_i = \frac{1}{i!} R^i w \qquad (0 \le i \le d)$$
$$\in E_{r+i}^* W$$

Then
$$Rw_i = (i+1) w_{i+1} \qquad (0 \le i \le d) \qquad (Rw_{-1} = 0 \cdot w_0, \text{ valid with } i=-1)$$
$$Rw_d = 0 \qquad \text{(by definition of } d\text{)}$$

<u>Claim</u>
$$Lw_0 = 0 \qquad \text{(If we set } w_{-1}=0\text{, this holds for } i=0\text{)}$$
$$Lw_i = (D-2r-i+1) w_{i-1} \qquad (1 \le i \le d)$$

(Pf) By induction on i,
The case $i=0$ is trivial and the case $i=1$ follows from above claim.
Let $i \ge 2$

$$Lw_i = \frac{LRw_{i-1}}{i} = \frac{(A^* + RL)w_{i-1}}{i} \qquad \text{(by Claim (a))}$$

(by induction hypothesis)
$$= \frac{1}{i}\Big((D-2(r+i-1)) w_{i-1} + (D-2r-(i-1)+1) \underbrace{Rw_{i-2}}_{(i-1)w_{i-1}} \Big)$$
$$= \frac{1}{i}\Big(\cancel{D}\cancel{-2r}\cancel{-2i}\cancel{+2} + iD \cancel{-D} - 2ri \cancel{+2r} - i^2 \cancel{+2i}\cancel{-1} + i \cancel{-1} \Big) w_i$$

$$= (D-2r-i+1)\, w_{i-1}.$$

Claim $w_0, \cdots, w_d$ is a basis for W.

(Pf) Let $W' = \mathrm{Span}\{w_0, \cdots, w_d\}$

Then W' is R, L invariant.

So it is $A = R+L$ invariant.

Also it is E_i^*-invariant for every i.

Hence W' is a T-module.

Since W is irreducible $W' = W$.

As w_i's are orthogonal,

they are linearly independent.

Note that $w_i \neq 0$ by the definition of d and Lemma 9 (iv)

Claim $d = D-2r$

(Pf)
$$\begin{aligned}
0 &= (LR - RL - A^*)\, w_d \qquad (\text{by (a)}) \\
&= 0 - (D-2r-d+1) R w_{d-1} - (D-2(r+d))\, w_d \\
&= -d(D-2r-d+1)\, w_d - (D-2(r+d))\, w_d \\
&= (-dD + 2rd + d^2 - d - D + 2r + 2d)\, w_d \\
&= (d^2 + (2r - D + 1) d + 2r - D)\, w_d \\
&= (d + 2r - D)(d+1)\, w_d
\end{aligned}$$

Hence $d = D-2r$.

Therefore with respect to a basis $w_0, w_1, \cdots, w_d$,

$A = L + R \qquad w_{-1} = w_{d+1} = 0$

$Lw_i = (d-i+1)\, w_{i-1}$.

$Rw_i = (i+1)\, w_{i+1}$

$$L = \begin{pmatrix} 0 & d & & & 0 \\ & 0 & d-1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix} \qquad R = \begin{bmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 2 & \ddots & & \\ & & \ddots & & \\ 0 & & & d & 0 \end{bmatrix}$$

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[Proof of Thm. 10 Continued]

(iii) Let $r = r'$.

$w_0, \cdots, w_d$ a basis for W with $w_i \in E_i^* W$

$w_0', \cdots, w_{d'}'$ a basis for W' with $w_i' \in E_i^* W'$

Then $d = D - 2r = D - 2r' = d'$

and
$$\sigma : W \longrightarrow W'$$
$$w_i \longrightarrow w_i'$$
is an isomorphism of T-modules by (i)

If $r \neq r'$, then
$$d = D - 2r \neq D - 2r' = d',$$
hence $\dim W \neq \dim W'$.

(iv) Let W_i be the irreducible T-module with end point i. Then
$$\dim E_r^* V = \binom{D}{r} = \sum_{i=0}^{r} \mathrm{mult}(W_i)$$
Hence we have that
$$\mathrm{mult}(W_r) = \binom{D}{r} - \binom{D}{r-1}.$$
by induction on r.

Theorem 11 Let $\Gamma = H(D,2)$. $D \geq 1$

Fix $x \in X$ Write

$$E_i^* \equiv E_i^*(x), \quad T \equiv T(x) \quad \text{and} \quad A^* \equiv \sum_{i=0}^{D} (D-2i) E_i^*$$

Let W be an irreducible T-module with endpoint r $(0 \leq r \leq D/2)$

Then

(i) W has a basis

$w_0^*, w_1^*, \ldots, w_d^*$ $\quad (d = D-2r)$

s.t. $w_i^* \in E_{i+r} W$ $\quad (0 \leq i \leq d)$

with respect to which the matrix representing A^* is

$$\begin{pmatrix} 0 & d & & & & \\ 1 & 0 & d-1 & & \text{\huge 0} & \\ & 2 & 0 & & & \\ & & & \ddots & 2 & \\ & \text{\huge 0} & & d-1 & 0 & 1 \\ & & & & d & 0 \end{pmatrix}$$

In particular

(ii) $E_i A^* E_j = 0 \quad$ if $|i-j| \neq 1 \quad (0 \leq i,j \leq D)$

Proof. We use the notation.

$$[\alpha, \beta] = \alpha\beta - \beta\alpha \; (= -[\beta, \alpha])$$

Recall

(a) $[L, R] = A^*$

(b) $[A^*, L] = 2L$

(c) $[A^*, R] = -2R$.

and $A = L + R$.

Write (a) − (c) in terms of A, A^*.

$$[A, A^*] = [L, A^*] + [R, A^*]$$
$$= 2(R-L)$$

$$\begin{cases} R + L = A \\ R - L = [A, A^*]/2 \end{cases}$$

Hence

$$R = \frac{2A + [A, A^*]}{4} \qquad \text{and}$$

$$L = \frac{2A - [A, A^*]}{4}$$

Now (a), (b) become

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0 \qquad \text{—— (1)}$$

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0 \qquad \text{—— (2)}$$

(pf.) By (b)

$$2A - \underline{AA^*} + \overset{*}{A}A = 4L = 2[A^*, L]$$

$$= A^* \frac{2A - [A, A^*]}{2} - \frac{2A - [A, A^*]}{2} A^*$$

$$= A^*A - \underline{AA^*} + \frac{1}{2}(-A^*AA^* + A^{*2}A + AA^{*2} - A^*AA^*)$$

So

$$A^{*2}A - 2A^*AA^* + A^{*2}A - 4A = 0 \quad \cdots\cdots \quad (2)$$

By (a)

$$-16A^* = \Big[2A + [A, A^*],\ 2A - [A, A^*]\Big]$$

$$= (2A + [A, A^*])(2A - [A, A^*]) - (2A - [A, A^*])(2A + [A, A^*])$$

$$= \cancel{4A^2} - 2A[A, A^*] + [A, A^*]2A - \cancel{[A, A^*]^2}$$

$$- \cancel{4A^2} - 2A[A, A^*] + [A, A^*]2A + \cancel{[A, A^*]^2}$$

$$= -4A^2A^* + 4AA^*A + 4AA^*A - 4A^*A^2$$

So

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0$$

Claim 1 $\quad E_i A^* E_j = 0 \quad$ if $\quad |i-j| \neq 1 \quad (0 \le i,j \le D)$

(pf.)

$$0 = E_i (A^2A^* - 2AA^*A + A^*A^2 - 4A^*) E_j$$
$$= E_i A^* E_j (\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4)$$
$$(AE_j = \theta_j E_j, \quad E_i A = (AE_i)^t = (\theta_i E_i)^t = \theta_i E_i)$$
$$= E_i A^* E_j (\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2)$$
$$= E_i A^* E_j (D-2i-(D-2j)-2)(D-2i-(D-2j)+2)$$
$$(\theta_k = D-2k)$$
$$= E_i A^* E_j \cdot 4 \cdot \underbrace{(i-j+1)}_{\neq 0}\underbrace{(i-j-1)}_{\neq 0}$$

Hence $\quad E_i A^* E_j = 0.$

Now define "dual raising matrix"

$$R^* := \sum_{i=0}^{D} E_{i+1} A^* E_i$$

So

$$R^* E_i V \subseteq E_{i+1} V \qquad (0 \le i \le D, \quad E_{D+1} V = 0)$$

Define "dual lowering matrix"

$$L^* = \sum_{i=0}^{D} E_{i-1} A^* E_i$$

Then

$$L^* E_i V \subseteq E_{i-1} V \qquad (0 \le i \le D, \quad E_{-1} V = 0)$$

Observe :

$$A^* = \left(\sum_{i=0}^{D} E_i\right) A^* \left(\sum_{j=0}^{D} E_j\right)$$
$$= L^* + R^* \qquad (3)$$

by Claim 1

<u>Claim 2</u>

(a) $[L^*, R^*] = A$

(b) $[A, L^*] = 2L^*$

(c) $[A, R^*] = -2R^*$

(pf)

(b). $$AL^* - L^*A = \sum_{i=0}^{D} \left(\underbrace{AE_{i-1}}_{\theta_{i-1}E_{i-1}} A^* E_i - E_{i-1} A^* \underbrace{E_i A}_{\theta_i E_i} \right)$$

$$= \sum_{i=0}^{D} E_{i-1} A^* E_i \, (\theta_{i-1} - \theta_i)$$

$\theta_k = D - 2k \quad \therefore \theta_{i-1} - \theta_i = 2i - 2(i-1) = 2$

$$= 2L^*.$$

(c) Similar

$\boxed{\text{HS}}$

$$AR^* - R^*A = \sum_{i=0}^{D} (A E_{i+1} A^* E_i - E_{i+1} A^* E_i A)$$

$$= \sum_{i=0}^{D} E_{i+1} A^* E_i \, (\theta_{i+1} - \theta_i)$$

$$= -2R^*$$

(a) $$[A, A^*] = [A, L^*] + [A, R^*]$$
$$= 2(L^* - R^*) \quad \text{by (b) and (c).} \qquad (4)$$

Since $A^* = L^* + R^*$,

$$R^* = \frac{2A^* + [A^*, A]}{4}$$

$$L^* = \frac{2A^* - [A^*, A]}{4}$$

Now (a) is seen to be equivalent to (2) upon evalution

This proves Claim 2.

$\boxed{HS}$

$$[L^*, R^*] = \frac{1}{16}\Big((2A^* - [A^*, A])(2A^* + [A^*, A]) - (2A^* + [A^*, A])(2A^* - [A^*, A])\Big)$$

$$= \frac{1}{16}\Big(\cancel{4A^{*2}} + 2A^*[A^*, A] - [A^*, A]2A^* - \cancel{[A^*, A]^2} - \cancel{4A^{*2}} + 2A^*[A^*, A] - [A^*, A]2A^* + \cancel{[A^*, A]^2}\Big)$$

$$= \frac{1}{4}(A^{*2}A - 2A^*AA^* + AA^{*2})$$

$$= A \qquad \text{by (2)}$$

Now apply same argument as for (1), (2) of Thm 10 and observe A^* has $D+1$ distinct eigenvalues

So $A^* = \sum_{i=0}^{D} (D-2i)E_i^*$ generates $M^* = \mathrm{Span}(E_0^*, \dots, E_D^*)$

Hence $E_0, \dots, E_D, A^*$ generates T.

Take an irreducible T-module W with endpoint r. $(0 \le r \le D/2)$.

Set $t = \min\{i \mid E_i W \ne 0\}$

Pick $0 \ne w_0^* \in E_t W$.

Set $w_i^* = \frac{1}{i!} R^{*i} w_0^* \in E_{t+i} W \quad \forall i$

Then $R^* w_i^* = (i+1) w_{i+1}^* \quad \forall i$

By (a) we get by induction $L^* w_i^* = (D-2t-i+1) w_{i-1}^*$

$$L^* w_i^* = \frac{1}{i} L^* R^* w_{i-1}^* = \frac{1}{i}(A + R^* L^*) w_{i-1}^*$$

$$= \frac{1}{i}\Big((D - 2(t+i-1)) w_{i-1}^* + (i-1)(D - 2t - i + 2) w_{i-1}^*\Big)$$

$iD - D - 2it + \cancel{t} - i^2 + i + \cancel{2i} - \cancel{2}$

$$= (D - 2t - i + 1) w_{i-1}^*$$

So $\mathrm{Span}(w_0^*, w_1^*, \dots)$ is L^*, R^*, A^*-invariant.

Hence $W = \mathrm{Span}(w_0^*, w_1^*, \dots, w_d^*)$

$w_0^*, w_1^*, \dots, w_d^* \ne 0, \quad w_i^* = 0 \quad \forall i > d$ by dimension

Thus $d = D - 2t$.

$(\because)$ $(D-2(t+d))\,w_d^* = A w_d^* = (L^*R^* - R^*L^*)\,w_d^*$

$= -(D-2t-d+1)\,R^* w_{d-1}^*$

$= -(D-2t-d+1)\,d\,w_d^*$

$d^2 + (2t-D-1+2)\,d - (D-2t) = 0$

$(d-D+2t)(d+1) = 0$ So $d = D-2t$

DEF. Any graph $\Gamma = (X, E)$

Pick $x \in X$. $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$

(i) an irreducible T-module W is <u>thin</u>, if $\dim E_i^* W \leq 1$ $\forall i$.

(ii) Γ is <u>thin with respect to</u> x, if every irreducible $T(x)$-module is thin

(iii) an irreducible T-module W is dual <u>thin</u> if $\dim E_i W \leq 1$ $\forall i$

(iv) Γ is <u>dual thin</u> with respect to x, if every irreducible $T(x)$-module is dual thin.

<u>Observe</u> : $H(D,2)$ is thin, dual thin with respect to each $x \in X$.

DEF. With above notation, write $D \equiv D(x)$.

(i) an ordering $E_0, E_1, \cdots, E_R$ of primitive idempotents of Γ is <u>restricted</u> if E_0 corresponds to the maximal eigenvalue.

Fix a restricted ordering
(ii) Γ is Q-polynomial with respect to x, above ordering if there exists
$A^* \equiv A^*(x) \in T$ s.t.
(a) $E_0^*V, \ldots, E_D^*V$ are the maximal eigenspaces for A^*
(b) $E_i A^* E_j = 0$ if $|i-j| > 1$ $(0 \le i, j \le R)$

Observe $H(D,2)$ is Q-polynomial with respect to natural ordering of the idempotents, and every vertex.

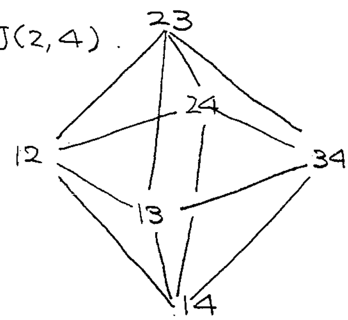
PROGRAM Study graphs that are thin and Q-polynomial with respect to each vertex.
(In fact, thin w.r.t. x $\rightarrow$ dual thin w.r.t. x)
Get a situation like $H(D,2)$,
where T is generated by A, A^*.
Except $sl_2(\mathbb{C})$ is replaced by a quantum Lie algebra

Lecture 7 Wed. Feb. 3, 1993

DEF. The Johnson graph $\Gamma = J(D, N)$ $(1 \le D \le N-1)$ satisfies

$$X = \{ S \mid S \subseteq \Omega, \ |S| = D \} \quad (\Omega = \{1, 2, \cdots, N\})$$
$$E = \{ ST \mid S, T \in X, \ |S \cap T| = D-1 \}$$

Example $J(2,4)$.



Note 1. The symmetric group S_N acts on Ω
$S_N \subseteq \mathrm{Aut}(\Gamma)$ acts vertex transitively on Γ

Note 2 $J(D, N)$ is isomorphic to $J(N-D, N)$.

$$\begin{array}{ccc} J(D,N) & & J(N-D,N) \\ \| & & \| \\ \Gamma = (X, E) & & \Gamma' = (X', E') \\ X \ni S & \longrightarrow & \bar{S} = \Omega \setminus S \in X' \end{array}$$

This correspondence induces an isomorphism of graphs

$$\left[\begin{array}{l} ST \in E \Leftrightarrow |S \cap T| = D-1 \\ \qquad\quad \Leftrightarrow |\Omega - (S \cup T)| = N-D-1 \\ \qquad\quad \Leftrightarrow |\bar{S} \cap \bar{T}| = N-D-1 \\ \qquad\quad \Leftrightarrow \bar{S}\bar{T} \in E' \end{array} \right]$$

Hence without loss of generality assume $D \le N/2$ for $J(D, N)$.

We will need the eigenvalues of $J(D,N)$ for certain problem later in course.
We can get these eigenvalues from our study of $H(N,2)$.

LEMMA 12 The eigenvalues for $J(D,N)$ $(1 \leq D \leq N/2)$ are given by

$$\theta_i = (N-D-i)(D-i)-i \quad , \quad (0 \leq i \leq D)$$
$$m_i = \binom{N}{i} - \binom{N}{i-1} .$$

Proof Let

$$\Gamma_J \equiv J(D,N) = (X_J, E_J)$$
$$\Gamma_H \equiv H(N,2) = (X_H, E_H)$$

Set $x \equiv 11 \cdots 1 \in X_H$

Define
$$\tilde{\Gamma} \equiv (\tilde{X}, \tilde{E}) ,$$
where
$$\tilde{X} = \{ y \in X_H \mid \partial_H(x,y) = D \} \qquad \partial_H : \text{distance in } \Gamma_H$$
$$\tilde{E} = \{ yz \in X_H \mid \partial_H(y,z) = 2 \}$$

Observe.
$$\begin{array}{ccc} X_J & \longrightarrow & \tilde{X} \\ S & \longmapsto & \hat{S} \end{array} \quad , \quad \text{where } \hat{S} = a_1 \cdots a_N \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$$
induces an isomorphism of graphs $\Gamma_J \longrightarrow \tilde{\Gamma}$.

$$\left[\begin{array}{llll} (\because) & ST \in E_J & \Leftrightarrow & |S \cap T| = D-1 \\ & & \Leftrightarrow & \partial_H(\hat{S}, \hat{T}) = 2 \\ & & \Leftrightarrow & (\hat{S}, \hat{T}) \in \tilde{E} . \end{array} \right]$$

Identify Γ_J with $\tilde{\Gamma}$.
Then the standard module V_J of Γ_J become $\tilde{V} = E_D^* V_H$.
where V_H is the standard module of Γ_H, and $E_D^* \equiv E_D^*(x)$

Let R be the raising matrix with respect to x in Γ_H,
let L be the lowering matrix with respect to x in Γ_H

Recall

$$(RL - DE_D^*)\hat{S} = \sum_{T \in X_J \ |S \cap T| = D-1} \hat{T}.$$

by proof of Thm 10. (Lec 5-6).
Hence

$$\tilde{A} := RL - DE_D^* |_{\tilde{V}}$$

is the adjacency map in $\tilde{\Gamma}$.

To find eigenvalues of $\tilde{A}$, pick any irreducible $T(x)$-module W with the endpoint $r \le D$.
Then by Thm 10

$$\operatorname{diam}(W) = N - 2r.$$

Let

$$w_0, w_1, \dots, w_{N-2r}$$

denote a basis for W as in Thm 10.
Then $w_{D-r} \in E_D^* W \subseteq \tilde{V}$

Observe: $\tilde{A} w_{D-r} = RL w_{D-r} - DE_D^* w_{D-r}$
$= R(N-2r-D+r+1) w_{D-r-1} - D w_{D-r}$
$= ((N-D-r+1)(D-r) - D) w_{D-r}$.
(This is valid for $D=r$ as well.)

Hence.

$$\tilde{A}\, w_{D-r} = \big((N-D-r)(D-r) - r\big)\, w_{D-r}$$

Let $\quad V_H = \sum W \qquad$ (direct sum of irreducible $T(x)$-modules)

Then

$$\begin{aligned} V_J &= E_D^* V_H \\ &= \sum_{W\,:\, r(W) \le D} E_D^* W \\ &= \text{a direct sum of 1 dimensional eigenspaces for } \tilde{A} \end{aligned}$$

The eigenspace for eigenvalue.

$(N-D-r)(D-r) - r \qquad \leftarrow$ (monotonously decreasing wrt r.)

appears with multiplicity.

$$\binom{N}{r} - \binom{N}{r-1}$$

in this sum by Thm 10 (iv) (Lec 5-3)

Theorem 13. Any graph $\Gamma = (X, E)$. Fix $x \in X$

$E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$ $D \equiv D(x)$, $K = \mathbb{C}$

Then we have the following implications of conditions

$$TH \Leftrightarrow C \Leftarrow S \Leftarrow G.$$

(TH) Γ is thin with respect to x.

(C) $E_i^* T E_i^*$ is commutative $\forall i$ $(0 \le i \le D)$

(S) $E_i^* T E_i^*$ is symmetric $\forall i$ $(0 \le i \le D)$

(G) $\forall y, z \in X$ with $\partial(x,y) = \partial(x,z)$
$\exists g \in \mathrm{Aut}\,\Gamma$ s.t. $gx = x$; $gy = z$, $gz = y$.

(Proof.)

(TH) $\Rightarrow$ (C) Fix i $(0 \le i \le D)$.

Let $V = \sum W$. (The standard module written as a direct sum of irreducible T-modules)

Then $E_i^* V = \sum E_i^* W$. (The direct sum of 1 dimensional $E_i^* T E_i^*$ modules.)

Since $\dim E_i^* W = 1$,

for $a, b \in E_i^* T E_i^*$, $ab - ba \,|\, E_i^* W = 0$.

Hence $ab - ba = 0$

(C) $\Rightarrow$ (TH) Suppose $\dim E_i^* W \ge 2$

for some irreducible T-module W with some i. $(0 \le i \le D)$

<u>Claim 1</u> $E_i^* W$ is an irreducible $E_i^* T E_i^*$-module:

(pf) Suppose

$0 \subsetneq U \subsetneq E_i^* W$,

where U is a $E_i^* T E_i^*$-module.

Then by the irreducibility

$TU = W$.

So $U \supseteq E_i^* T E_i^* U = E_i^* T U = E_i^* W$

This is a contradiction

Claim 2. Each irreducible $S = E_i^* T E_i^*$ module U has dimension 1. In particular Γ is thin w.r.t. x.

(pf.) Pick

$$0 \neq a \in E_i^* T E_i^*$$

Since $\mathbb{C}$ is algebraically closed, a has an eigenvector $w \in U$ with eigenvalue θ.

Then

$$\begin{aligned}(a - \theta I) U &= (a - \theta I) S w \\ &= S (a - \theta I) w \\ &= 0.\end{aligned}$$

Hence

$$a|_U = \theta I|_U \qquad \text{for } \forall a \in S.$$

Thus, each 1 dimensional subspace of U is an S-module.

We have

$$\dim U = 1$$

By Claim 1 & Claim 2, we have (TH).

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[Proof of Thm 13 continued]

(S) ⇒ (C).

Fix i and pick $a, b \in E_i^* T E_i^*$

Since a, b and ab are symmetric,

$$ab = (ab)^t = b^t a^t = ba$$

Hence $E_i^* T E_i^*$ is commutative.

(G) ⇒ (S)

Fix i and pick $a \in E_i^* T E_i^*$.

Pick $y, z \in X$.

We want to show that

$$a_{yz} = a_{zy}.$$

We may assume that

$$\partial(x, y) = \partial(x, z) = i.$$

otherwise

$$a_{yz} = a_{zy} = 0 .$$

By our assumption, there exists $g \in G$ such that

$$g(y) = z, \quad g(z) = y \quad g(x) = x.$$

Let $\hat{g}$ denote the permutation matrix representing g, i.e.,

$$\hat{g}\hat{y} = \widehat{g(y)} \qquad \forall y \in X \qquad \hat{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \leftarrow y$$

If $g \in \mathrm{Aut}\,\Gamma$, then

$$\hat{g} A = A \hat{g} \qquad \text{(Exercise)}$$

Also we have

$$\hat{g} E_j^* = E_j^* \hat{g} \qquad (0 \le j \le D),$$

since

$$\partial(x, y) = \partial(g(x), g(y))$$
$$= \partial(x, g(y)).$$

Hence $\hat{g}$ commutes with each element of T.

We have

$$a_{yz} = (\hat{g}^{-1} a \hat{g})_{yz} \qquad (\hat{g})_{yz} = \begin{cases} 1 & g(z)=y \\ 0 & \text{else} \end{cases}$$

$$= \sum_{y', z'} (\hat{g}^{-1})_{yy'} a_{y'z'} \hat{g}_{z'z}$$

(zero except for $g^{-1}(y') = y \quad g(z) = z'$)

$$= a_{g(y)\, g(z)}$$

$$= a_{zy}$$

This proves Thm 13.

<u>Open Problem</u> :

Find all the graphs that satisfy the condition (G) for every vertex x.

($H(N,2)$ is one example.

$\because \quad \mathrm{Aut}\,\Gamma_{(1,\cdots,1)} \simeq S_\Omega \qquad x = (1 \cdots 1)$

$\Gamma_i(x) = \{ \hat{S} \mid |S| = i \}$)

Property (G) is clearly related to <u>distance-transitive</u> property.

DEF. Any graph $\Gamma = (X, E)$ with $G \le \mathrm{Aut}\,\Gamma$ is said to be <u>distance-transitive</u> (or two-point homogeneous), whenever for $\forall x, x', y, y' \in X$ with $\partial(x, y) = \partial(x', y')$ there exists $g \in G$ s.t.

$$g(x) = y \qquad g(x') = y'.$$

(This means G is as close to being doubly transitive as possible.)

LEMMA 14 Suppose a graph $\Gamma = (X, E)$ satisfies the property $(G) = (G(x))$ for every $x \in X$. Then

(i) either

(ia) Γ is vertex transitive ; or

(ib) Γ is bipartite $(X = X^+ \cup X^-)$ with X^+, X^- each an orbit of $Aut(\Gamma)$

(ii) if (ia) holds, then Γ is distance-transitive.

Proof. (i)

Claim Suppose $y, z \in X$ are connected by a path of even length. Then y, z are in the same orbit of $Aut(\Gamma)$.

(pf.) It suffices to assume that the path has length 2 $\quad y \sim w \sim z$.

Now $\partial(y, w) = \partial(w, z) = 1$. So there exists $g \in Aut(\Gamma)$ such that

$gw = w, \quad gy = z \quad gz = y$.

This proves Claim

Fix $x \in X$.

Now suppose Γ is not vertex transitive, and we shall show (ib)

Observe $X = X^+ \cup X^-$, where

$X^+ = \{ y \in X \mid \exists$ path of even length connecting x and $y \}$.

$X^- = \{ y \in X \mid \exists$ path of odd length connecting x and $y \}$.

Also X^+ is contained in an orbit O^+ of $Aut(\Gamma)$

X^- is contained in an orbit O^- of $Aut(\Gamma)$

Now $O^+ \cap O^- = \phi$ (else $O^+ = O^- = X$ and vertex Transitive)

So $X^+ = O^+, \quad X^- = O^-$

Also $X^+ \cup X^- = X$ is a bipartition by construction.

(ii) Fix x, y, x', y' with $\partial(x,y) = \partial(x',y')$
By vertex transitivity,
there exists an element
$g_1 \in G$ s.t. $g_1 x = x'$.

Observe $\partial(x',y') = \partial(x,y)$
$= \partial(g_1 x, g_1 y) = \partial(x', g_1 y)$

Hence there exists an element
$g_2 \in G$ s.t.
$g_2 x' = x'$ $g_2 y' = g_1 y$. $g_2 g_1 y = y'$.
by $(G(x'))$ property.

Set $g = g_2 g_1$. Then
$gx = x'$ $gy = y'$
by construction.

The following graphs $\Gamma = (X, E)$ are vertex transitive and satisfy the property $(G(x))$ for $\forall x \in X$.

$J(D, N)$, $H(D, r)$, $J_q(D, N)$

where

<u>$H(D,r)$</u>:
$X = \{ a_1 \cdots a_D \mid a_i \in F \quad 1 \le i \le D \}$
F : any set of cardinality r.
$E = \{ xy \mid x, y \in X$,
x, y differ in exactly one coordinate $\}$.

$J_q(D,N)$.

X = the set of all D-dimensional subspaces of N-dimensional vector space / $GF(q)$.

$E = \{ xy \mid x, y \in X,\ \dim(x \cap y) = D-1 \}$

The following graph is distance-transitive but does not satisfy $(G(x))$ for any $x \in G$.

$H_q(D,N)$:

X = the set of all $D \times N$ matrices with entries in $GF(q)$

$E = \{ xy \mid x, y \in X,\ \mathrm{rank}(x-y) = 1 \}$

$\boxed{\text{HS}}$ $H(D,r)$ $\quad G = S_r \text{ wr } S_D \qquad G_x = S_{r-1} \text{ wr } S_D$

$\partial(x,y) = \partial(x,z) = i$

$Y = \{ j \in \Omega \mid x_j \neq y_j \} \leftrightarrow Z = \{ j \in \Omega \mid x_j \neq z_j \}$

$(y_{j_1} \cdots y_{j_i}) \leftrightarrow (z_{l_1} \cdots z_{l_i})$

$J(D,N)$ $\quad G = S_N \qquad G_x = S_D \times S_{N-D}$

$X \cap Y \leftrightarrow X \cap Z$

$(\Omega - X) \cap Y \leftrightarrow (\Omega - X) \cap Z$

$J_q(D,N)$

$X \cap Y \leftrightarrow X \cap Z$

The theory of a single thin irreducible T-module

$\Gamma = (X, E)$ any graph

M = Bose-Mesner algebra / $K = \mathbb{C}$
generated by the adjacency matrix A.
$= \mathrm{Span}(E_0, \cdots, E_R)$.

M acts on the standard module. $V = \mathbb{C}^{|X|}$.
Fix $x \in X$
$D \equiv D(x)$ is x-diameter
$k = k(x)$ is the valency of x.

Lecture 9 Mon. Feb 8 . 1993

Let $\Gamma = (X, E)$ be any graph.

M : Bose-Mesner algebra over $K = \mathbb{C}$ generated by the adjacency matrix A

$M = \mathrm{Span}(E_0, E_1, \cdots, E_R)$

M acts on the standard module $V = \mathbb{C}^{|X|}$.

Fix $x \in X$.

$D = D(x)$: x-diameter

$k = k(x)$: valency of x

DEF. Pick $x \in X$ Write $E_i^* = E_i^*(x)$, $T \equiv T(x)$

Let W be an irreducible thin T-module with endpoint r, diameter d.

Let $a_i = a_i(W) \in \mathbb{C}$ satisfying

$$E_{r+i}^* A E_{r+i}^* \big|_{E_{r+i}^* W} = a_i 1 \big|_{E_{r+i}^* W} \quad (0 \le i \le d)$$

Let $x_i = x_i(W) \in \mathbb{C}$ satisfying

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^* \big|_{E_{r+i-1}^* W} = x_i 1 \big|_{E_{r+i-1}^* W} \quad (1 \le i \le d)$$

LEMMA 15 With above notation, the following hold.

(i) $a_i \in \mathbb{R}$ $(0 \le i \le d)$

(ii) $x_i \in \mathbb{R}^{>0}$ $(1 \le i \le d)$

(iii) Pick $0 \ne w_0 \in E_r^* W$. Set $w_i = E_{r+i}^* A^i w_0$ $\forall i$ then.

(iii a) $w_0, w_1, \cdots, w_d$ is a basis for W, $w_{-1} = w_{d+1} = 0$.

(iii b) $A w_i = w_{i+1} + a_i w_i + x_i w_{i-1}$ $(0 \le i \le d)$,

(iv) Define $p_0, p_1, \cdots, p_{d+1} \in \mathbb{R}[\lambda]$ by

$p_0 = 1$, $\lambda p_i = p_{i+1} + a_i p_i + x_i p_{i-1}$ $(0 \le i \le d)$ $p_{-1} = 0$. then

(iv a) $p_i(A) w_0 = w_i$ $(0 \le i \le d+1)$

(iv b) p_{d+1} is the minimal polynomial of $A|W$.

Proof (i) a_i is an eigenvalue of a real symmetric matrix $E^*_{r+i} A E^*_{r+i}$.

(ii) x_i is an eigenvalue of a real symmetric matrix $B^t B$, where

$$B = E^*_{r+i} A E^*_{r+i-1}.$$

Hence $x_i \in \mathbb{R}$.

Since $B^t B$ is positive semidefinite.

$x_i \geq 0$.

(pf If $B^t B v = \sigma v$ for some $\sigma \in \mathbb{R}$, $v \in \mathbb{R}^m - \{0\}$ then

$$0 \leq \|Bv\|^2 = v^t B^t B v = \sigma v^t v = \sigma \|v\|^2 \qquad \|v\|^2 > 0.$$

So $\sigma \geq 0$)

Moreover.

$x_i \neq 0$, by Lemma 9 (iv).

(iiia) Observe.

$$w_i = E^*_{r+i} A E^*_{r+i-1} w_{i-1} \qquad (1 \leq i \leq d)$$

So $w_i \neq 0$ $(0 \leq i \leq d)$ by Lemma 9 (iv)

Hence

$$W = \mathrm{Span}(w_0, \cdots, w_d) \qquad \text{by Lemma 9 (iii).}$$

(iiib) We have that

$$A w_i = E^*_{r+i+1} A w_i + E^*_{r+i} A w_i + E^*_{r+i-1} A w_i$$

$$= w_{i+1} + E^*_{r+i} A E^*_{r+i} w_i + E^*_{r+i-1} A E^*_{r+i} A E^*_{r+i-1} w_{i-1}$$

$$= w_{i+1} + a_i w_i + x_i w_{i-1}$$

(iva) Clear for $i = 0$., Assume OK for $0, \cdots, i$.

$$p_{i+1}(A) w_0 = (A - a_i I) w_i - x_i w_{i-1} = w_{i+1}$$

(ivb) By definition,

$p_{d+1}(A) w_0 = 0$

Moreover $p_{d+1}(A) W = 0$:

For $\forall w \in W$, write

$$w = \sum_{i=0}^{d} \alpha_i w_i = \sum_{i=0}^{d} \alpha_i p_i(A) w_0 \qquad \text{some } \alpha_i \in \mathbb{C}.$$

$$= p(A) w_0 \qquad \text{some } p \in \mathbb{C}[\lambda]$$

Hence

$$p_{d+1}(A) w = p_{d+1}(A) p(A) w_0$$
$$= p(A) p_{d+1}(A) w_0$$
$$= 0.$$

p_{d+1} is the minimal polynomial :

Suppose. $q(A) W = 0$

for some $0 \neq q \in \mathbb{C}[\lambda]$. $\deg q < \deg p_{d+1} = d+1$.

Then

$$q = \sum_{i=0}^{d} \beta_i p_i \qquad \text{for some } \beta_i \in \mathbb{C}.$$

We have

$$0 = q(A) w_0 = \sum_{i=0}^{d} \beta_i p_i(A) w_0$$
$$= \sum_{i=0}^{d} \beta_i w_i$$

Hence $\beta_0 = \cdots = \beta_d = 0$ by (iii a).

Thus $q = 0$,

a contradiction

Cor 16. Let Γ, W, r, d be as above.
Then
(i) W is dual thin. (i.e.,
$\dim E_i W \leq 1 \quad (0 \leq i \leq R)$)
(ii) $d = |\{i \mid E_i W \neq 0\}| - 1$.

Proof (i) Set as in Lemma 15

$$w_i = p_i(A) w_0 \in E^*_{r+i} W$$

Then $w_0, w_1, \cdots, w_d$ is a basis for W.
We have

$$W = M w_0$$

So. $E_i W = E_i M w_0 = \operatorname{Span} E_i w_0$

Thus.

$$\dim E_i W = \begin{cases} 1 & \text{if} \quad E_i w_0 \neq 0 \\ 0 & \text{if} \quad E_i w_0 = 0 \end{cases}$$

In particular,
$\dim E_i W \leq 1$.

(ii) Immediate as
$\dim W = d + 1$.

Note. Γ, W, r, d as above.
$E^*_r W = \operatorname{Span}\{v\}$ $\quad W = Tv$.
By (ii) and Lemma 15, $W = Mv = \operatorname{Span}\{v_0, v_1, \ldots, v_d\}$

LEMMA 17 Given an irreducible thin $T(x)$-module W with endpoint $r = r(W)$, diameter $d = d(W)$

Write $x_i = x_i(W)$ $(0 \le i \le d)$

$w_i = p_i(A) w_0$ $(0 \le i \le d)$ $0 \ne w_0 \in E_r^* W$

$\in E_{r+i}^* W$

Then

$$\frac{\| w_i \|^2}{\| w_0 \|^2} = x_1 x_2 \cdots x_i \qquad (1 \le i \le d)$$

Proof It suffices to show that

$$\| w_i \|^2 = x_i \| w_{i-1} \|^2 \qquad (1 \le i \le d)$$

Recall by Lemma 15 (iii b) that

$$A w_j = w_{j+1} + a_j w_j + x_j w_{j-1} \qquad (0 \le j \le d)$$

$$(w_{-1} = w_{d+1} = 0)$$

Now observe

$$\langle w_{i-1}, A w_i \rangle = \langle w_{i-1}, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle$$

$$= \overline{x_i} \| w_{i-1} \|^2$$

$$= x_i \| w_{i-1} \|^2$$

by Lemma 15 (ii)

Also

$$\langle w_{i-1}, A w_i \rangle = \langle A w_{i-1}, w_i \rangle \qquad (\text{since } \overline{A}^t = A)$$

$$= \langle w_i + a_{i-1} w_{i-1} + x_{i-1} w_{i-2}, w_i \rangle$$

$$= \| w_i \|^2$$

DEF Let W be an irreducible thin $T(x)$-module with endpoint r, $E_i^* \equiv E_i^*(x)$.
The *measure* $m = m_W$ is the function
$m : \mathbb{R} \longrightarrow \mathbb{R}$.
such that

$$m(\theta) = \begin{cases} \dfrac{\|E_i w\|^2}{\|w\|^2}, & \text{where } 0 \neq w \in E_r^* W \\ & \text{if } \theta = \theta_i \text{ is an eigenvalue for } \Gamma \\ 0 & \text{if } \theta \text{ is not an eigenvalue for } \Gamma. \end{cases}$$

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$\Gamma = (X, E)$ any graph.
Fix $x \in X$.
$E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$ subconstituent algebra $/\mathbb{C}$
$V = \mathbb{C}^{|X|}$: standard module.

LEMMA 18. With above notation let W denote a thin irreducible $T(x)$-module with endpoint r diameter d.

Let
$a_i = a_i(W)$ $(0 \le i \le d)$:
$x_i = x_i(W)$ $(1 \le i \le d)$
$p_i = p_i(W)$ $(0 \le i \le d+1)$
be from Lemma 15, and
measure $m \equiv m_W$.

Then

(i) $p_0, \dots, p_{d+1}$ are orthogonal with respect to m, i.e.,

$$\sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \qquad (0 \le i, j \le d+1) \quad x_{d+1} = 0$$

(ia) $$\sum_{\theta \in \mathbb{R}} p_i^2(\theta) m(\theta) = x_1 \cdots x_i \qquad (0 \le i \le d)$$

(ib) $$\sum_{\theta \in \mathbb{R}} m(\theta) = 1$$

(ii) $$\sum_{\theta \in \mathbb{R}} p_i^2(\theta)\, \theta\, m(\theta) = x_1 \cdots x_i a_i \qquad (0 \le i \le d)$$

Proof Pick $0 \ne w_0 \in E_r^* W$.
Set $w_i = p_i(A) w_0 \in E_{r+i} W$

Since $E_i^* W$ and $E_j^* W$ are orthogonal $(i \ne j)$

$$\delta_{ij} \| w_i \|^2 = \langle w_i, w_j \rangle$$
$$= \langle p_i(A) w_0, p_j(A) w_0 \rangle$$

$$= \left\langle p_i(A)\left(\sum_{l=0}^{R} E_l\right) w_0,\quad p_j(A)\left(\sum_{l=0}^{R} E_l\right) w_0 \right\rangle$$

$$= \left\langle \sum_{l=0}^{R} p_i(\theta_l) E_l w_0,\quad \sum_{l=0}^{R} p_j(\theta_l) E_l w_0 \right\rangle \qquad (\text{as } AE_j = \theta_j E_j)$$

$$= \sum_{l=0}^{R} p_i(\theta_l)\,\overline{p_j(\theta_l)}\,\|E_l w_0\|^2 \qquad (\langle E_l w_0, E_{l'} w_0\rangle = 0 \quad l \neq l')$$

$$= \sum_{l=0}^{R} p_i(\theta_l)\, p_j(\theta_l)\, m(\theta_l) \|w_0\|^2 \qquad \left(\begin{array}{l} p_j \in \mathbb{R}[\lambda],\ \theta_l \in \mathbb{R} \\ m(\theta_i)\|w_0\|^2 = \|E_i w_0\|^2 \end{array} \right)$$

$$= \sum_{\theta \in \mathbb{R}} p_i(\theta)\, p_j(\theta)\, m(\theta) \|w_0\|^2.$$

Now done by Lemma 17 as

$\|w_i\|^2 = \|w_0\|^2 x_1 x_2 \cdots x_i$

For (ia) set $i = j$.

For (ib) set $i = j = 0$

(ii) $\langle w_i, Aw_i\rangle = \langle w_i, w_{i+1} + a_i w_i + x_i w_{i-1}\rangle$

$$= \bar{a}_i \|w_i\|^2$$

$$= a_i x_1 \cdots x_i \|w_0\|^2 \qquad a_i \in \mathbb{R} \text{ by Lemma 15}$$

Also

$$\langle w_i, Aw_i\rangle = \langle p_i(A) w_0,\ A p_i(A) w_0\rangle$$

$$= \left\langle p_i(A)\left(\sum_{l=0}^{R} E_l\right) w_0,\ A p_i(A)\left(\sum_{l=0}^{R} E_l\right) w_0 \right\rangle \quad \text{as in (i)}$$

$$= \sum_{l=0}^{R} p_i(\theta_l)^2\, \theta_l\, \|E_l w_0\|^2$$

$$= \sum_{\theta \in \mathbb{R}} p_i(\theta)^2\, \theta\, m(\theta) \|w_0\|^2$$

Thus we have ii)

Lemma 19. With above notation let W be a thin irreducible $T(x)$-module with measure m.

Then m determines

diameter $d(W)$,

$a_i = a_i(W)$ $\quad (0 \le i \le d)$

$x_i = x_i(W)$ $\quad (1 \le i \le d)$

$p_i = p_i(W)$ $\quad (0 \le i \le d+1)$

Proof. $d+1 = \#$ of $\theta \in \mathbb{R}$ s.t. $m(\theta) \neq 0$.

Hence m determines d.

Apply (ia) (ii) of Lemma 18

$$\sum_{\theta \in \mathbb{R}} m(\theta) = 1 \qquad p_0 = 1$$

$$\sum_{\theta \in \mathbb{R}} \theta\, m(\theta) = a_0 \qquad p_1 = \lambda - a_0$$

$$\left.\begin{aligned} &\sum_{\theta \in \mathbb{R}} p_1(\theta)^2 m(\theta) = x_1 \\ &\sum_{\theta \in \mathbb{R}} p_1(\theta)^2 \theta\, m(\theta) = x_1 a_1 \to a_1 \end{aligned}\right\} \Rightarrow p_2 = (\lambda - a_1) p_1 - x_1 p_0$$

$$\left.\begin{aligned} &\sum_{\theta \in \mathbb{R}} p_2(\theta)^2 m(\theta) = x_1 x_2 \to x_2 \\ &\sum_{\theta \in \mathbb{R}} p_2(\theta)^2 \theta\, m(\theta) = x_1 x_2 a_2 \to a_2 \end{aligned}\right\} \Rightarrow p_3 = (\lambda - a_2) p_2 - x_2 p_1$$

$$\vdots$$

$$\left.\begin{aligned} &\sum_{\theta \in \mathbb{R}} p_d(\theta)^2 m(\theta) = x_1 x_2 \cdots x_d \to x_d \\ &\sum_{\theta \in \mathbb{R}} p_d(\theta)^2 \theta\, m(\theta) = x_1 x_2 \cdots x_d a_d \to a_d \end{aligned}\right\} \Rightarrow p_{d+1} = (\lambda - a_d) p_d - x_d p_{d-1}$$

COR. 20 With above notation, let W, W' denote thin irreducible T-modules.
The following are equivalent.

(i) W, W' are isomorphic as T-modules

(ii) $r(W) = r(W')$, $m_W = m_{W'}$

(iii) $r(W) = r(W')$, $d(W) = d(W')$, $a_i(W) = a_i(W')$ $(0 \le i \le d)$
$x_i(W) = x_i(W')$ $(1 \le i \le d)$.

Proof (i) ⇒ (iii)

Write $r = r(W)$, $r' = r(W')$, $d = d(W)$, $d' = d(W')$
$a_i = a_i(W)$, $a_i' = a_i(W')$, $x_i = x_i(W)$, $x_i' = x_i(W')$.

Let $\sigma : W \longrightarrow W'$

denote an isomorphism of T-modules

For $\forall i$

$$\sigma E_i^* W = E_i^* \sigma W = E_i^* W'$$

So

$$r = r' \qquad d = d'.$$

To show $a_i = a_i'$:

Pick $w \in E_{r+i}^* W - \{0\}$.

Then

$$E_{r+i}^* A E_{r+i}^* \sigma(w) = \sigma(E_{r+i}^* A E_{r+i}^* w)$$

$$= \sigma(a_i w) = a_i \sigma(w)$$

$\sigma w \neq 0$. So

$a_i =$ eigenvalue of $E_{r+i}^* A E_{r+i}^*$ on $E_{r+i}^* W$.

$= a_i'$

To show $x_i = x_i'$: similar.

[$\boxed{HS}$ Pick $w \in E_{r+i-1}^* W - \{0\}$]

$$E^*_{r+i-1} A E^*_{r+i} A E^*_{r+i-1} \sigma(\omega) = \sigma\left(E^*_{r+i-1} A E^*_{r+i} A E^*_{r+i-1} \omega\right) = x_i \sigma(\omega)$$

x_i = eigenvalue of $E^*_{r+i-1} A E^*_{r+i} A E^*_{r+i-1}$ on $E^*_{r+i-1} W = x_i'$

(iii) $\Rightarrow$ (i)

Pick $0 \neq w_0 \in E_r^* W$ $\quad 0 \neq w_0' \in E_r^* W'$.

Set $w_i = p_i(A) w_0 \in E^*_{r+i} W \quad (0 \le i \le d)$ $\quad p_i$ from Lemma 15

$w_i' = p_i'(A) w_0' \in E^*_{r+i} W' \quad (0 \le i \le d)$

Define linear transformation

$\sigma : W \to W' \quad (w_i \to w_i')$

Since $\{w_i\}$, $\{w_i'\}$ are bases with $d = d'$,
σ is an isomorphism of vector spaces.

We need to show

$$a\sigma = \sigma a \quad \forall a \in T \qquad \text{— } (*)$$

Take $a = E_j^*$ some j $\quad (0 \le j \le d(x))$

$\forall i$

$$\begin{array}{ccc} E_j^* \sigma w_i & \stackrel{?}{=} & \sigma E_j^* w_i \\ \| & & \| \\ E_j^* w_i' & & \delta_{ij} \sigma(w_i) \\ \| & & \| \\ \delta_{ij} w_i' & & \delta_{ij} w_i' \end{array}$$

Take a = adjacency matrix A

$$\begin{array}{ccc} A\sigma w_i & \stackrel{?}{=} & \sigma A w_i \\ \| & & \| \\ A w_i' & & \sigma(w_{i+1} + a_i w_i + x_i w_{i-1}) \\ \| & & \| \\ & w_{i+1}' + a_i' w_i' + x_i' w_{i-1}' & \end{array}$$

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(ii) $\Rightarrow$ (iii) Lemma 19

(iii) $\Rightarrow$ (ii)

Given d, a_i, x_i. we can compute the polynomial sequence

$$p_0, p_1, \cdots, p_{d+1}$$

for W.

Show $p_0, p_1, \ldots, p_{d+1}$ determines $m = m_W$.

Set

$$\Delta = \{ \theta \in \mathbb{R} \mid p_{d+1}(\theta) = 0 \}$$

☆ Observe ; $|\Delta| = d+1$ ← See 'An Introduction to Interlacing'

$m(\theta) = 0$ if $\theta \notin \Delta$ $(\theta \in \mathbb{R})$

So it suffices to find $m(\theta)$ $(\theta \in \Delta)$

By Lemma 18 (i)

$$\left.\begin{aligned} \sum_{\theta\in\Delta} m(\theta)\, p_0(\theta) &= 1 \\ \sum_{\theta\in\Delta} m(\theta)\, p_1(\theta) &= 0 \\ &\vdots \\ \sum_{\theta\in\Delta} m(\theta)\, p_d(\theta) &= 0 \end{aligned}\right\}$$

$d+1$ linear equations with $d+1$ unknowns $m(\theta)$ $(\theta\in\Delta)$

But the coefficient matrix is essencially Van der Monde (since $\deg p_i = i$).

Hence the system is nonsingular and there are unique values for $m(\theta)$ $(\theta\in\Delta)$

$\boxed{\text{HS}}$ ☆ $\begin{pmatrix} \theta-a_0 & -1 & & \\ -x_1 & \theta-a_1 & -1 & \\ & \ddots & \ddots & -1 \\ & & -x_d & \theta-a_d \end{pmatrix} \begin{pmatrix} p_0(\theta) \\ \vdots \\ p_d(\theta) \end{pmatrix} = 0$

θ: eigen values of $\begin{pmatrix} a_0 & 1 & & 0 \\ x_1 & a_1 & 1 & \\ & & \ddots & 1 \\ 0 & & x_d & a_d \end{pmatrix}$ ← diagonalizable

mult. of θ

$= \dim \mathrm{Ker}(\theta I - L) = 1$

Lecture 11 Fri. Feb 12, 1993

$\Gamma = (X, E)$: connected graph

θ_0 : maximal eigenvalue of Γ

δ : corresponding eigenvector

$$\delta = \sum_{y \in X} \delta_y \hat{y} \qquad (\text{WLOG } \delta_y \in \mathbb{R}^{>0} \text{ for } \forall y \in X)$$

LEMMA 21. Fix $x \in X$. Write $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$.

(i) $T\delta = T\hat{x}$ is an irreducible T-module.

(ii) Given any irreducible T-module W, the following are equivalent:

(iia) $W = T\delta$

(iib) The diameter $d(W) = d(x)$.

(iic) The endpoint $r(W) = 0$

PROOF. (i) Observe : $\exists$ irreducible T-module W that contains δ:

Standard module $V = \sum W_i$ (direct sum of irreducible T-modules)

$\operatorname{Span}\delta = E_0V = \sum E_0W_i$.

So $E_0W_i \neq 0$ for some i.

Then $\delta \in E_0W_i \subset W_i$.

Observe : $T\delta$ is an irreducible T-module :

$\delta \in W$: irreducible T-module

$T\delta \leq W$. Since W is irreducible $T\delta = W$.

Observe : $T\delta = T\hat{x}$:

$\hat{x} = \delta_x^{-1} E_0^* \delta \in T\delta$. So $T\hat{x} \subset T\delta$

Since $T\delta$ is irreducible $T\hat{x} = T\delta$.

(ii) (a) $\to$ (b) :

$$E_i^* \delta = \sum_{y \in X \ \partial(x,y)=i} \delta_y \hat{y} \neq 0 \qquad (0 \leq i \leq d(x))$$

because $\delta_y > 0$ for every $y \in X$.

Hence $E_i^* T \delta \neq 0 \qquad (0 \leq i \leq d(x))$

Thus $d(x) = d(W)$.

(b) $\to$ (c) :

Immediate

(c) $\to$ (a) :

Since $r(W) = 0$, $E_0^* W \neq 0$.

Hence $\hat{x} \in W$ and $T\hat{x} \subset W$.

By the irreducibility, we have $T\hat{x} = W$.

DEF The irreducible $T(x)$-module $T(x)\delta$ in Lemma 21 is called the trivial $T(x)$-module.

LEMMA 22 Assume Γ is <u>bipartite</u> $(X = X^+ \cup X^-)$ (nonempty)

Then the following are equivalent.

(i) $\exists\, \alpha^+, \exists\, \alpha^- \in \mathbb{R}$ s.t.

$$\delta_x = \begin{cases} \alpha^+ & \text{if } x \in X^+ \\ \alpha^- & \text{if } x \in X^- \end{cases}$$

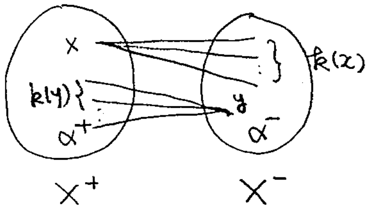
(ii) $\exists\, k^+, \exists\, k^- \in \mathbb{Z}^{>0}$ s.t.

$$\text{valency } k(x) = \begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^- \end{cases}$$

In this case $k^+ k^- = \theta_0^2$, and

Γ is called <u>bi-regular</u>.

PROOF. (i) $\to$ (ii)



$$A\delta = A\Big(\alpha^+ \sum_{x\in X^+} \hat{x} + \alpha^- \sum_{y \in X^-} \hat{y}\Big)$$

$$= \alpha^+ \sum_{y\in X^-} k(y)\,\hat{y} + \alpha^- \sum_{x\in X^+} k(x)\,\hat{x}$$

$$= \theta_0\, \delta$$

So $\quad k(x)\,\alpha^- = \theta_0 \alpha^+$

$\quad k(y)\,\alpha^+ = \theta_0 \alpha^-$

As $\alpha^+ \neq 0$, $\alpha^- \neq 0$,

$k^+ := k(x)$ is independent of the choice of $x \in X^+$ and

$k^- := k(y)$ is independent of the choice of $y \in X^-$

Moreover $k^+ k^- = \theta_0^2$

(ii) $\to$ (i)

Set $\delta' = \sum_{y\in X} \alpha_y \hat{y}$ where $\alpha_y = \begin{cases} 1/\sqrt{k^-} & \text{if } y \in X^+ \\ 1/\sqrt{k^+} & \text{if } y \in X^- \end{cases}$

Then one checks

$$A\delta' = A\Big(\frac{1}{\sqrt{k^-}} \sum_{y\in X^+} \hat{y} + \frac{1}{\sqrt{k^+}} \sum_{y\in X^-} \hat{y}\Big) = \frac{k^-}{\sqrt{k^-}} \sum_{y \in X^-} \hat{y} + \frac{k^+}{\sqrt{k^+}} \sum_{y\in X^+} \hat{y} = \sqrt{k^+ k^-}\, \delta'$$

Since $\delta' > 0$, $\delta' \in \mathrm{Span}(\delta)$
and $\theta_0 = \sqrt{k^+ k^-}$.

DEF. For any graph $\Gamma = (X, E)$, fix $x \in X$.
Set $d = d(x)$.
Γ is <u>distance-regular</u> <u>w.r.t. x</u> if $\forall i$ $(0 \leq i \leq d)$
and all $y \in X$ s.t. $\partial(x, y) = i$;

$$c_i(x) := |\{z \in X \mid \partial(x,z) = i-1,\ \partial(y,z) = 1\}|$$

$$a_i(x) := |\{z \in X \mid \partial(x,z) = i,\ \partial(y,z) = 1\}|$$

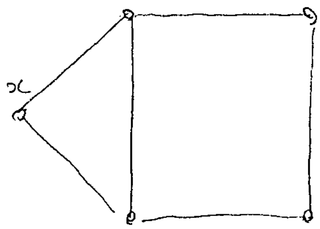
$$b_i(x) := |\{z \in X \mid \partial(x,z) = i+1,\ \partial(y,z) = 1\}|$$

depends only on i, x, and not on y.

(In this case $c_0(x) = a_0(x) = b_d(x) = 0$
$c_1(x) = 1$.
$b_0(x) = k(x)$ is the valency of x)

We call $c_i(x)$, $a_i(x)$, $b_i(x)$, the intersection numbers w.r.t x.

<u>An example</u>



$c_0 = 0$	$c_1 = 1$	$c_2 = 1$
$a_0 = 0$	$a_1 = 1$	$a_2 = 1$
$b_0 = 2$	$b_1 = 1$	$b_2 = 0$

Lecture 12 Mon. Feb 15. 1993

LEMMA 23. For any connected graph $\Gamma = (X, E)$, the following are equivalent

(i) The trivial $T(x)$-module is thin for $\forall x \in X$

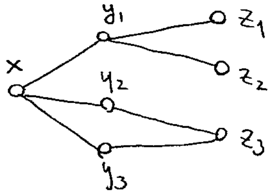
(ii) $\left\{ \sum_{\substack{y \in X \\ \partial(x,y)=i}} \hat{y} \;\middle|\; 0 \le i \le d(x) \right\}$

is a basis for the trivial $T(x)$-module for $\forall x \in X$.

(iii) Γ is distance-regular w.r.t. x for $\forall x \in X$.

If (i)-(iii) holds, then Γ is regular or biregular.

Note. (i), (ii) are not equivalent for single x.



$E_0^* T\hat{x} = \langle \hat{x} \rangle \qquad E_1^* T\hat{x} = \langle y_1 + y_2 + y_3 \rangle$

$E_2^* T\hat{x} = \langle z_1 + z_2 + 2z_3 \rangle$

PROOF OF LEMMA 23

(i) $\Rightarrow$ (ii)

Let $\delta = \sum_{y \in X} \delta_y \hat{y}$ be an eigenvector for the maximal eigenvalue θ_0.

$$\sum_{\substack{y \in X \\ \partial(x,y)=1}} \hat{y} = A\hat{x} \in T(x)\hat{x} = T(x)\delta \ni E_1^*\delta = \sum_{\substack{y \in X \\ \partial(x,y)=1}} \delta_y \hat{y}$$

If the trivial $T(x)$-module is thin,

$\delta_y = \delta_z$ for $y, z \in X$ $\partial(x,y) = \partial(x,z) = 1$.

Hence

$\delta_y = \delta_z$ if y and z in X are connected by a path of even length

So Γ is regular or bipartite biregular by LEMMA 22

In particular $\delta_y = \delta_z$ if $\partial(x,y) = \partial(x,z)$, as there is a path of length $2\cdot\partial(x,y)$
$y \sim \cdots \sim x \sim \cdots \sim z$

Hence $E_i^*\delta \in \mathrm{Span}\left(\sum_{\substack{y\in X\\ \partial(x,y)=i}} \hat{y}\right)$.

Since $E_0^*\delta, E_1^*\delta, \cdots, E_d^*\delta$ form a basis for $T(x)\delta$, we have (ii).

(ii) → (iii)

Fix $x \in X$ $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$ $d \equiv d(x)$.

$$A \sum_{\substack{y\in X\\ \partial(x,y)=i}} \hat{y} = \sum_{z\in X} \left|\{y\in X \mid \partial(y,z)=1,\ \partial(x,y)=i\}\right| \hat{z}$$

$$= \sum_{\substack{z\in X\\ \partial(x,y)=i-1}} b_{i-1}(x,z)\hat{z} + \sum_{\substack{z\in X\\ \partial(x,z)=i}} a_i(x,z)\hat{z} + \sum_{\substack{z\in X\\ \partial(x,z)=i+1}} c_{i+1}(x,z)\hat{z}$$

$$\in \mathrm{Span}\left\{ \sum_{\substack{z\in X\\ \partial(x,z)=j}} \hat{z} \;\middle|\; j = 0,1,\cdots,d \right\}$$

Hence $b_{i-1}(x,z)$, $a_i(x,z)$, $c_{i+1}(x,z)$ depend only i, x and not on z.
Therefore Γ is distance-regular w.r.t. x.

(iii) → (i) Fix $x\in X$, $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, $d = d(x)$.
By definition of distance-regularity, for $\forall i$ $(0 \le i \le d)$,

$$A\Big(\sum_{y\in X\ \partial(x,y)=i} \hat{y}\Big) = b_{i-1}(x)\sum_{y\in X\ \partial(x,y)=i-1} \hat{y} + a_i(x)\sum_{y\in X,\partial(x,y)=i} \hat{y} + c_{i+1}(x)\sum_{y\in X\ \partial(x,y)=i+1} \hat{y}$$

Hence $W = \left\{ \sum_{\substack{y\in X\\ \partial(x,y)=i}} \hat{y} \;\middle|\; 0\le i\le d \right\}$ is A-inv and so T-inv. Since $\hat{x} \in W$. $T\hat{x} = W$ is the trivial module and $T\hat{x}$ is thin.

Next, we show more is true if (i) - (iii) hold in Lemma 23.

In fact $d(x)$, $a_i(x)$, $c_i(x)$, $b_i(x)$ are

$$\begin{cases} \text{independent of } x & \text{if } \Gamma \text{ is regular} \\ \text{constant over } X^+, X^- & \text{if } \Gamma \text{ is biregular} \end{cases}$$

Let $\Gamma = (X, E)$ be any (connected) graph.
Pick $x, y \in X$

Let W be a thin, irreducible $T(x)$-module
Let W' be a thin, irreducible $T(y)$-module
measure $m : \mathbb{R} \to \mathbb{R}$ determined by W
measure $m' : \mathbb{R} \to \mathbb{R}$ determined by W'

Recall W, W' are <u>orthogonal</u> if
$\langle w, w' \rangle = 0 \quad \forall w \in W, \ w' \in W'$.

We will show if W, W' are <u>not</u> orthogonal,
then m, m' are related:

$$m \cdot \text{poly}_1 = m' \cdot \text{poly}_2$$

for some polynomials with

$$\deg \text{poly}_1 + \deg \text{poly}_2 \leq 2\partial(x, y)$$

<u>Notation</u>

V : standard module of Γ.
H : any subspace of V
$V = H + H^{\perp}$ (orthogonal direct sum)
$v = v_1 + v_2$

$$\text{proj}_H : V \longrightarrow H$$
$$v \longrightarrow v_1 \quad : \text{linear transformation.}$$

Observe: $\forall v \in V$

$v - \mathrm{proj}_H v \in H^\perp$

So

$\langle v - \mathrm{proj}_H v, h \rangle = 0 \quad \forall h \in H$ or

$\langle v, h \rangle = \langle \mathrm{proj}_H v, h \rangle \quad (\forall v \in V, \forall h \in H)$.

THEOREM 24 $\Gamma = (X, E)$ any graph

Pick $x, y \in X$. Set $\Delta = \partial(x, y)$.

Assume

W: thin irreducible $T(x)$-module, with end point r, diameter d, measure m.

W': thin irreducible $T(y)$-module with end point r', diameter d', measure m'.

W, W' are not orthogonal

Now pick

$0 \neq w \in E_r^*(x) W$

$0 \neq w' \in E_{r'}^*(x) W'$

Then

(i) $\mathrm{proj}_{W'} w = p(A) \dfrac{\|w\|}{\|w'\|} w'$ for some $0 \neq p \in \mathbb{C}[\lambda]$ with $\deg p \leq \Delta - r' + r, d'$

$\mathrm{proj}_W w' = p'(A) \dfrac{\|w'\|}{\|w\|} w$ for some $0 \neq p' \in \mathbb{C}[\lambda]$ with $\deg p' \leq \Delta - r + r', d$.

(ii) $\dfrac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = m(\theta_i) \overline{p'(\theta_i)}$

$= m'(\theta_i) p(\theta_i)$

for all eigenvalues θ_i of Γ.

(iii) For all eigenvalues θ_i of Γ.

$p(\theta_i) p'(\theta_i)$

is a real number in interval $[0, 1]$

Proof

(i) Since W, W' are not orthogonal,
$\exists v \in W$, $\exists v' \in W'$ s.t. $\langle v, v' \rangle \neq 0$.
Then there exists $a \in M$ such that
$v' = a w'$.
($\because$ $w_i' = p_i'(A) w_0'$ $\therefore$ $\forall v' \in W'$ $\exists g \in \mathbb{C}[\lambda]$, $g(A) w_0' = v'$)
We have

$$0 \neq \langle v', v \rangle = \langle a w', v \rangle = \langle w', \underset{W}{\underset{\Uparrow}{a^* v}} \rangle$$

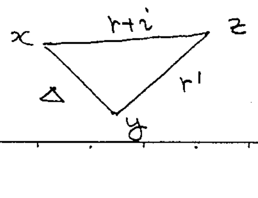
Hence $\quad \mathrm{proj}_W\, w' \neq 0$.

Let $p_0, \cdots, p_d \in \mathbb{C}[\lambda]$ be from Lemma 15.
Then
$w_i = p_i(A) w$ is a basis for $E^*_{r+i}(x) W$ $(0 \le i \le d)$
Hence
$\mathrm{proj}_W\, w' = \alpha_0 w_0 + \cdots + \alpha_d w_d$ some $\alpha_j \in \mathbb{C}$.
Set

$$p' := \frac{\|w\|}{\|w'\|} \sum_{i=0}^{d} \alpha_i p_i$$

Then $\quad 0 \neq p' \in \mathbb{C}[\lambda]$, $\quad \deg p' \le d$

<u>Claim</u> $\quad \alpha_i = 0 \quad (\Delta - r + r' < i \le d)$
In particular, $\deg p' \le \Delta - r + r'$
($\because$)
Observe $\quad w' \in E^*_{r'}(y) V$
$\quad\quad\quad\quad w \in E^*_r(x) V$
for $\partial(x, y) = \Delta$
$E^*_{r'}(y) V \cap E^*_{r+i}(x) V = 0$

$\nexists$ 

by triangle inequality.

Hence

$$E^*_{r'}(y)V \perp E^*_{r+i}(x)V$$

or

$$0 = \langle \omega', \omega_i \rangle$$
$$= \langle \mathrm{proj}_W \omega', \omega_i \rangle$$
$$= \sum_{j=0}^{d} \alpha_j \langle \omega_j, \omega_i \rangle$$
$$= \alpha_i \|\omega_i\|^2$$

Hence

$$\alpha_i = 0$$

Thus

$$\mathrm{proj}_W \omega' = \sum_{i=0}^{\Delta+r'-r} \alpha_i \omega_i$$
$$= \sum_{i=0}^{\Delta+r'-r} \alpha_i p_i(A)\omega_0$$
$$= p'(A) \frac{\|\omega'\|}{\|\omega\|} \omega.$$

(ii) $$\frac{\langle E_i\omega, E_i\omega' \rangle}{\|\omega\|\|\omega'\|} = \frac{\langle E_i\omega, \omega' \rangle}{\|\omega\|\|\omega'\|}$$
$$= \frac{\langle E_i\omega, \mathrm{proj}_W \omega' \rangle}{\|\omega\|\|\omega'\|}$$
$$= \frac{\langle E_i\omega, p'(A)\omega \rangle}{\|\omega\|^2}$$
$$= \frac{\langle E_i\omega, E_i p'(A)\omega \rangle}{\|\omega\|^2}$$
$$= \overline{p'(\theta_i)} \frac{\|E_i\omega\|^2}{\|\omega\|^2}$$
$$= \overline{p'(\theta_i)}\, m(\theta_i)$$

$\mathrm{proj}_W \omega' = p'(A)\frac{\|\omega'\|}{\|\omega\|}\omega$

$$\frac{\langle E_i\omega, E_i\omega'\rangle}{\|\omega\|\|\omega'\|} = \frac{\overline{\langle E_i\omega', E_i\omega\rangle}}{\|\omega'\|\|\omega\|} = \overline{\overline{p(\theta_i)}\, m'(\theta_i)}$$

$$= p(\theta_i)\, m'(\theta_i)$$

as $m(\theta_i),\ m'(\theta_i) \in \mathbb{R}$

(iii) $$\frac{|\langle E_i\omega, E_i\omega'\rangle|^2}{\|\omega\|^2\|\omega'\|^2} = p(\theta_i)\,p'(\theta_i)\,m(\theta_i)\,m'(\theta_i)$$

$$\therefore\ p(\theta_i)\,p'(\theta_i) = \frac{|\langle E_i\omega, E_i\omega'\rangle|^2}{m(\theta_i)\,m'(\theta_i)\ \|\omega\|^2\|\omega'\|^2} \in \mathbb{R}$$

$$= \frac{|\langle E_i\omega, E_i\omega'\rangle|^2}{\frac{\|E_i\omega\|^2}{\|\omega\|^2}\frac{\|E_i\omega'\|^2}{\|\omega'\|^2}\ \|\omega\|^2\|\omega'\|^2}$$

By Cauchy-Schwartz inequality
($|\langle a, b\rangle| \leq \|a\|\,\|b\|$.)

$$\frac{|\langle E_i\omega, E_i\omega'\rangle|^2}{\|E_i\omega\|^2\,\|E_i\omega'\|^2} \leq 1$$

Hence we have the assertion.

o.

ate

Lecture 13 Wed. Feb. 17, 1993

LEMMA 25 $\Gamma = (X, E)$ any graph.

Pick an edge $xy \in E$.

Assume the trivial $T(x)$-module $T(x)\delta$ is thin with measure m_x.

and the trivial $T(y)$-module $T(y)\delta$ is thin with measure m_y

Then

(ia) $\dfrac{m_x(\theta)}{k_x} = \dfrac{m_y(\theta)}{k_y}$ $\forall \theta \in \mathbb{R} - \{0\}$

$k_x \leftarrow$ valency of x

(ib) $\dfrac{m_x(0) - 1}{k_x} = \dfrac{m_y(0) - 1}{k_y}$

($\delta = \sum_{y \in X} \delta_y \hat{y}$ eigenvector corresponding to the maximal eigenvalue)

Proof.

Apply Thm 24.

$W = T(x)\delta$ $r = 0$ $d = d(x)$

$W' = T(y)\delta$ $r' = 0$ $d' = d(y)$.

Take $w = \hat{x}$, $w' = \hat{y}$.

Claim $\operatorname{proj}_{T(y)\delta} \hat{x} = k_y^{-1} A \hat{y}$.

(pf.) Since

$\hat{y} \in T(y)\delta$, $A\hat{y} \in T(y)\delta$

Show $\left(\hat{x} - \frac{1}{k_y} A\hat{y}\right) \perp \left(T(y)\delta\right)$

Recall $A\hat{y} = \sum_{z \in X,\ yz \in E} \hat{z}$

$\hat{x} - \frac{1}{k_y} A\hat{y} \in E_1^*(y) V$

So

$$\hat{x} - \frac{1}{k_y} A\hat{y} \quad \perp \quad E_j^*(y)\, T(y)\, \delta \qquad \forall\ j \neq 1 \quad (0 \le j \le d(y))$$

And we have

$$\langle \hat{x} - \frac{1}{k_y} A\hat{y},\ A\hat{y} \rangle$$

$$= \langle \hat{x}, \sum_{\substack{z \in X \\ yz \in E}} \hat{z} \rangle - \frac{1}{k_y} \| \sum_{\substack{z \in X \\ yz \in E}} \hat{z} \|^2$$

$$= 1 - 1$$

$$= 0.$$

This proves Claim

Similarly

$$\operatorname{proj}_{T(x)\delta}' \hat{y} = k_x^{-1} A\hat{x}$$

Hence the polynomial $p, p' \in \mathbb{C}[\lambda]$ from Thm 24 equals $\frac{\lambda}{k_y}$ and $\frac{\lambda}{k_x}$ respectively.

By Thm 24

$$m_x(\theta)\, \overline{p'(\theta)} = m_y(\theta)\, p(\theta)$$

$$\frac{m_x(\theta)\,\theta}{k_x} \qquad \frac{m_y(\theta)\,\theta}{k_y}$$

If $\theta \neq 0$, we have (ia)

Also $\frac{1 - m_x(0)}{k_x} = \left(\sum_{\theta \in \mathbb{R} - \{0\}} m_x(\theta) \right) \frac{1}{k_x}$ by (ia)

$$= \left(\sum_{\theta \in \mathbb{R} - \{0\}} m_y(\theta) \right) \frac{1}{k_y} = \frac{1 - m_y(0)}{k_y}$$

Hence we have (ib)

THEOREM 26 Suppose any graph $\Gamma = (X, E)$ is distance-regular wrt x $\forall x \in X$
(So Γ is regular or bi-regular by Lemma 23.)
Then
<u>Case</u> Γ reg : the diameter $d(x)$ and the intersection numbers $a_i(x), b_i(x), c_i(x)$ $(0 \le i \le d(x))$ are independent of $x \in X$.
(And Γ is called <u>distance-regular</u>)
<u>Case</u> Γ : bireg : $(X = X^+ \cup X^-)$
$d(x)$ and $a_i(x), b_i(x), c_i(x)$ $(0 \le i \le d(x))$ are constant over X^+, X^-
(and Γ is called <u>distance-biregular</u>)

Proof. By Lemma 25.
<u>Case</u> Γ : regular
Then $m_x = m_y$ $\forall xy \in E$.
Hence the measure of the trivial $T(x)$-module is independent of $x \in X$.

Case Γ : biregular
Then $m_x = m_{x'}$ $\forall x, x' \in X$ with $\partial(x, x') = 2$.
Hence the measure of the trivial $T(x)$-module is constant over $x \in X^+, X^-$.

Fix $x \in X$. Write $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, $W = T\delta$ with mesure m, diameter $d = d(x)$.
We know, by COR 20 that m determines d, $a_i(W)$, $(0 \le i \le d)$, $x_i(W)$ $(1 \le i \le d)$
(as $d = D(x) = d(W)$ by Lemma 21)
We shall show that m determines $a_i(x)$, $c_i(x)$, $b_i(x)$ $(0 \le i \le d)$

Observe : $a_i(W) = a_i(x) \qquad (0 \le i \le d)$

$x_i(W) = b_{i-1}(x)\, c_i(x) \qquad (1 \le i \le d)$

$\boxed{\text{HS}}$ $a_i = a_i(W)$ is an eigenvalue of $E_i^* A E_i^*$ on $E_i^* W = \langle \sum_{y \in \Gamma_i(x)} \hat{y} \rangle$

(See Lemma 23)

$x_i = x_i(W)$ is an eigenvalue of $E_{i-1}^* A E_i^* A E_{i-1}^*$ on $E_{i-1}^* W$

and

$$A \sum_{y \in X\ \partial(x,y)=i} \hat{y} = b_{i-1}(x) \sum_{y \in X\ \partial(x,y)=i-1} \hat{y} + a_i(x) \sum_{y \in X,\, \partial(x,y)=i} \hat{y} + c_{i+1}(x) \sum_{y \in X\, \partial(x,y)=i+1} \hat{y}$$

So $x_i = b_{i-1}(x)\, c_i(x)$

Set $k^+ = k_x$

Define $k^- = \dfrac{\theta_0^2}{k^+}$ $\qquad$ (θ_0 = maximal eigenvalue)
[See Lemma 22]

(So $k^+ = k^-$ = valency if Γ is regular).

$\forall i \ (0 \le i \le d) \qquad \forall z \in X. \quad \partial(x, z) = i$

$$k_z = c_i(x) + a_i(x) + b_i(x) = \begin{cases} k^+ & : \text{if } i : \text{even} \\ k^- & \text{if } i : \text{odd.} \end{cases}$$

Now m determines

$c_0(x) = a_0(x) = 0, \qquad c_1(x) = 1.$

$b_0(x) = b_0(x)\, c_1(x) = x_1(W).$

$k^+ = b_0(x).$

$k^- = \theta_0^2 / k^+$

$c_i(x) = x_i(W) / b_{i-1}(x). \qquad (1 \le i \le d).$

$$b_i(x) = \begin{cases} k^+ - a_i(x) - c_i(x) & i : \text{even} \\ k^- - a_i(x) - c_i(x) & i : \text{odd} \end{cases} \qquad (1 \le i \le d)$$

PROPOSITION 27 Under the assumption of THM 26.

Case Γ : regular

(i) $\dim E_i V = |X| m(\theta_i)$

(ii) Γ has exactly $d+1$ distinct eigenvalues

$(d = \operatorname{diam}\Gamma = d(x) \quad \forall x \in X)$

Case Γ : biregular.

(i) $\dim E_i V = |X^+| m^+(\theta_i) + |X^-| m^-(\theta_i)$

(ii) Γ has exactly $d^+ + 1$ distinct eigenvalues

$(d^+ \geq d^-)$.

(iii) If d^+ is odd, the Γ is regular.

(iv) $d^+ = d^-$ or

$d^+ = d^- + 1$ is even.

(v) $a_i(x) = 0 \quad \forall i \quad \forall x$

PROOF. (i).

Γ: reg.

m_x: measure of the trivial $T(x)$-module.

$m_x(\theta_i) = \| E_i \hat{x} \|^2$

as $\|\hat{x}\| = 1$

Now

$$
\begin{aligned}
|X| m_x(\theta_i) &= \sum_{x \in X} m_x(\theta_i) \\
&= \sum_{x \in X} \| E_i \hat{x} \|^2 \\
&= \sum_{y,z} |(E_i)_{yz}|^2 \\
&= \operatorname{trace} E_i \overline{E_i}^t
\end{aligned}
$$

Since A is real symmetric,

E_i is real symmetric and

$E_i \overline{E_i}^t = E_i^2 = E_i$

with E_i symmetric.

$$E_i \sim \left(\begin{array}{cc|c} 1 & & \\ & 1 & O \\ \hline & O & O \end{array}\right)$$

trace E_i = rank E_i = dim E_iV

Thus we have the assertion in this case

Γ : bireg

same except

$$\sum_{x \in X} m_x(\theta_i) = |X^+|\, m^+(\theta_i) + |X^-|\, m^-(\theta_i)$$

(ii)

Γ : reg : Immediately if θ is an eigenvalue of Γ then $m(\theta) \neq 0$.

Γ : bireg : $\forall\, \theta = \theta_i \in \mathbb{R} \setminus \{0\}$

$$m^-(\theta) \neq 0 \Leftrightarrow m^+(\theta) \neq 0$$

$\Leftrightarrow$ θ : eigenvalue of Γ

$$\left(\frac{m^+(\theta)}{k^+} = \frac{m^-(\theta)}{k^-} \right)$$

(iv) & (v) Clear

$\boxed{\text{HS}}$ (iii) . If d^+ is odd, $d^+ = d^-$ and Γ has even number of eigenvalues, i.e., 0 is not an eigenvalue

So A is nonsingular and Γ is regular

Lecture 14 Fri. Feb. 19, 1993

Summary

Def. Assume $\Gamma = (X, E)$ is distance-regular w.r.t. x $\forall x \in X$.

Notation : $\forall x \in X$

	Case DR	Case DBR
valency k_x	k	k^+ if $x \in X^+$ k^- if $x \in X^-$
x-diameter D_x	D	D^+ if $x \in X^+$ D^- if $x \in X^-$
measure m_x of triv. $T(x)$-mod.	m	m^+ if $x \in X^+$ m^- if $x \in X^-$
int. number $c_i(x)$	c_i	c_i^+ if $x \in X^+$ c_i^- if $x \in X^-$
int. number $b_i(x)$	b_i	b_i^+ if $x \in X^+$ b_i^- if $x \in X^-$
int. number $a_i(x)$	a_i	0

Call m, $m^{\pm}$ the measure of Γ.

Assume $\Gamma = (X, E)$ is distance-regular.

To what extent do a_i's, b_i's c_i's determine structure of irreducible $T(x)$-modules. In general

LEMMA 28 Assume $\Gamma = (X, E)$ is distance-regular. Pick $x \in X$. Let W be a thin irreducible $T(x)$-module with endpoint r, diameter d, measure m_W.

(i) There is a unique polynomial $f_W \in \mathbb{C}[\lambda]$ s.t.

(ia) $\deg f_W \leq D$ (diam of Γ)

(ib) $m_W(\theta) = m(\theta) f_W(\theta) \quad \forall \theta \in \mathbb{R}$

m: measure of Γ

Moreover $f_W \in \mathbb{R}[\lambda]$.

(ii) $\deg f_W \leq 2r$.

(iii) For all eigenvalues θ_i of Γ, $\lambda - \theta_i$ is a factor of f_W whenever $E_i W = 0$. In particular, $2r - D + d \geq 0$.

Proof.

Let $\theta_0, \dots, \theta_D$ denote distinct eigenvalues of Γ.

$m(\theta_i) \neq 0 \quad (0 \leq i \leq D)$ by PROP 27

There exists unique $f_W \in \mathbb{C}[\lambda]$ with $\deg f_W \leq D$ s.t.

$$f_W(\theta_i) = \frac{m_W(\theta_i)}{m(\theta_i)} \qquad (0 \leq i \leq D)$$

by polynomial interpolation

$f_W \in \mathbb{R}[\lambda]$

since $\theta_0, \dots, \theta_D \in \mathbb{R}$ and $f_W(\theta_0), \dots, f_W(\theta_D) \in \mathbb{R}$

(ii) WLOG. $r < D/2$ else trivial.
Pick $0 \neq \omega \in E_r^*(x) W$

$$\omega = \sum_{\substack{y \in X \\ \partial(x,y) = r}} \alpha_y \hat{y} \qquad \text{some } \alpha_y \in \mathbb{C}$$

Pick $y \in X$ s.t. $\alpha_y \neq 0$

Set $W' =$ triv. $T(y)$-module $\qquad \left[\begin{array}{l} \langle \omega, \hat{y} \rangle \neq 0 \\ W \not\cong W' \end{array} \right]$
$r' = 0$. $m' = m$. $\Delta = r$.

Apply Thm 24.
$\deg p \leq \Delta - r' + r = 2r \qquad p \neq 0$
$\deg p' \leq \Delta - r + r' = 0 \qquad p' \neq 0$
$m_W(\theta) \overline{p'(\theta)} = m(\theta) p(\theta) \qquad (\forall \theta \in \mathbb{R})$

So

$\deg p/\overline{p}' \leq 2r$
and
$p/\overline{p}'$ satisfies the conditions of f_W.
$\left(\dfrac{p(\theta)}{\overline{p}(\theta)} = \dfrac{m_W(\theta)}{m(\theta)} \right)$

(iii) $E_i W = 0 \Rightarrow m_W(\theta_i) = 0 \Rightarrow f_W(\theta_i) = 0$
i.e., $E_i W = 0$
$\Rightarrow \theta_i$ is a root of $f_W(\lambda) = 0$.
So
$2r \geq \deg f_W \geq |\{\theta_i \mid E_i W = 0\}| = D - d$.
Hence
$2r - D + d \geq 0$.

No. Lec 14-4

Date 1995 · 2 · 9

LEMMA 29 Let $\Gamma = (X, E)$ be any distance-regular graph with valency k, diameter D ($d \geq 2$), measure m, eigenvalues

$$k = \theta_0 > \theta_1 > \cdots > \theta_D$$

Pick $x \in X$. Let W be a thin, irreducible $T(x)$-module with endpoint $r = 1$, diameter d, measure $m_W = m f_W$.

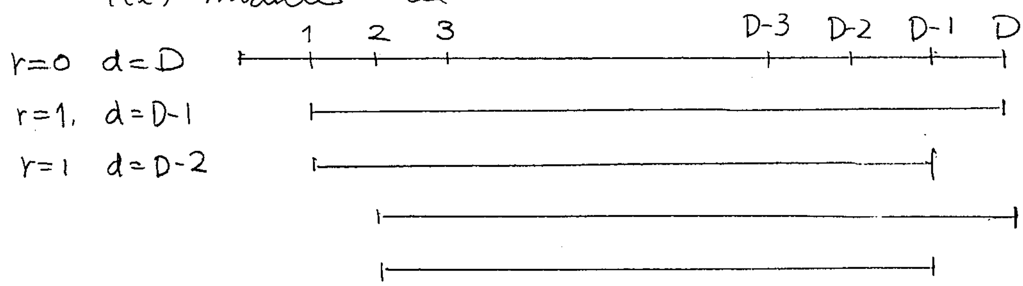
Then one of the following cases (i) - (4) occurs

Case	d	$f_W(\lambda)$	$a_0(W)$
(i)	$D-2$	$\dfrac{(\lambda-k)(\lambda-\theta_1)}{k(\theta_1+1)}$	$-\dfrac{b_1}{\theta_1+1}-1$
(ii)	$D-2$	$\dfrac{(\lambda-k)(\lambda-\theta_D)}{k(\theta_D+1)}$	$-\dfrac{b_1}{\theta_D+1}-1$
(iii)	$D-1$	$\dfrac{k-\lambda}{k}$	-1
(iv)	$D-1$	$\dfrac{(\lambda-k)(\lambda-\beta)}{k(\beta+1)}$	$-\dfrac{b_1}{\beta+1}-1$

for some $\beta \in \mathbb{R}$ with $\beta \in (-\infty, \theta_D) \cup (\theta_1, \infty)$.

Moreover, the isomorphism classs of W is determined by $a_0(W)$

Note By (3) the possible "shapes" of a thin irreducible $T(x)$-modules are



Lecture 15 Mon Feb. 22 1993

Proof of Lemma 29

We have $\deg f_W \leq 2$ by Lemma 28 (ii)
Also by Lemma 21 $E_0 W = 0$
(as otherwise $\langle \delta \rangle = E_0 V \subset W$ and $r(W) = 0$)
Hence $\lambda - \theta_0 = \lambda - k$
is a factor of f_W by Lemma 28 (iii)

Let $p_0, p_1, \cdots, p_D$ denote the polynomials for the trivial $T(x)$-module from Lemma 15.

Recall

$$\sum_{\theta \in \mathbb{R}} m(\theta)\, p_i(\theta)\, p_j(\theta) = \delta_{ij}\, x_1 x_2 \cdots x_i \qquad (0 \leq i, j \leq D)$$

$$= \delta_{ij}\, b_0 b_1 \cdots b_{i-1} c_1 c_2 \cdots c_i$$

($x_i = b_{i-1} c_i$ Lec13-4)
by Lemma 18.

By construction,

$p_0 = 1$.
$p_1(\lambda) = \lambda$
$p_2(\lambda) = \lambda^2 - a_1 \lambda - k$

Apparently,

$f_W = \sigma_0 p_0 + \sigma_1 p_1 + \sigma_2 p_2$
for some $\sigma_0, \sigma_1, \sigma_2 \in \mathbb{C}$.

Claim $\sigma_0 = 1$

$$\sigma_1 = \frac{a_0(W)}{k}$$

$$\sigma_2 = -\frac{(1 + a_0(W))}{k b_1}$$

No. Lec 15-2

Date 1995. 2. 10

Pf of Claim

$$1 = \sum_{\theta \in \mathbb{R}} m_W(\theta) \qquad \text{(by Lemma 18 (ib))}$$

$$= \sum_{\theta \in \mathbb{R}} m(\theta) f_W(\theta) \qquad \text{(by Lemma 28 (ib)}$$

$$= \sum_{j=0}^{2} \sigma_j \Big(\sum_{\theta \in \mathbb{R}} m(\theta)\, p_j(\theta) \Big)$$

$$= \sigma_0 \qquad \text{(by Lemma 18 (i))}$$

Next by Lemma 18 (ii) and $p_1(\theta) = \theta$

$$a_0(W) = \sum_{\theta \in \mathbb{R}} m_W(\theta)\, \theta$$

$$= \sum_{\theta \in \mathbb{R}} m(\theta) f_W(\theta)\, \theta$$

$$= \sum_{j=0}^{2} \sigma_j \sum_{\theta \in \mathbb{R}} m(\theta)\, p_j(\theta)\, p_1(\theta)$$

$$= \sigma_1 x_1(T\delta)$$

$$= \sigma_1 b_0 c_1$$

$$= \sigma_1 k$$

So far

$$f_W(\lambda) = 1 + \frac{a_0(W)}{k}\lambda + \sigma_2(\lambda^2 - a_1\lambda - k)$$

But

$$0 = f_W(k)$$

$$= 1 + a_0(W) + \sigma_2 k (k - a_1 - 1)$$

$$= 1 + a_0(W) + \sigma_2 k b_1$$

$$\sigma_2 = -\frac{(1 + a_0(W))}{k b_1}$$

This proves the claim