<u>Case</u> $a_0(W) = -1$

Here $\sigma_2 = 0$ and

$$f_W(x) = 1 + \frac{a_0(W)\lambda}{k}$$

$$= 1 - \frac{\lambda}{k}$$

Also

$$d+1 = |\{\theta \mid \theta : \text{eigenvalue of } \Gamma. \quad f_W(\theta) \neq 0\}|$$
$$= D.$$

<u>Case</u> $a_0(W) \neq -1$

Here $\sigma_2 \neq 0$ and

$$\deg f_W = 2$$

So

$$f_W(\lambda) = (\lambda - k)(\lambda - \beta)\alpha$$

for some $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$.

Comparing coefficients in

$$(\lambda - k)(\lambda - \beta)\alpha = 1 + \frac{a_0(W)}{k}\lambda - \frac{(a_0(W)+1)}{k b_1}(x^2 - a_1\lambda - k)$$

We find

$$\begin{cases} \alpha = -\dfrac{(a_0(W)+1)}{k b_1} \\ -(k+\beta)\alpha = \dfrac{a_0(W)}{k} + \dfrac{(a_0(W)+1)}{k b_1} a_1 \\ k\beta\alpha = 1 + \dfrac{a_0(W)+1}{b_1} \end{cases}$$

$$-\beta(a_0(W)+1) = b_1 + (a_0(W)+1)$$

$$(1 + a_0(W))(1+\beta) = -b_1 \qquad \text{—} \quad (*)$$

In particular, $\beta \neq -1$

and $$\alpha = -\frac{1+a_0(W)}{k b_1} = \frac{1}{k(\beta+1)}$$

Also

$$0 \leq m_W(\theta) \qquad \text{(Lec 9-6 DEF)}$$
$$= m(\theta) f_W(\theta) \qquad (\forall \theta \in \mathbb{R})$$

But if $\theta$ is an eigenvalue of $\Gamma$

$$0 < m(\theta).$$

So

$$0 \leq f_W(\theta) = \frac{(\theta - k)(\theta-\beta)}{k(\beta+1)}$$

Either

$$\beta + 1 > 0 \rightarrow \theta - \beta \leq 0 \quad \text{or} \quad \beta \geq \theta_1$$

or

$$\beta + 1 < 0 \rightarrow \theta - \beta \geq 0 \quad \text{or} \quad \beta \leq \theta_D$$

If $\beta = \theta_1$,

$$\left.\begin{aligned} a_0(W) &= -\frac{b_1}{\beta+1} - 1 = -\frac{b_1}{\theta_1+1} - 1 \\ f_W(\lambda) &= \frac{(\lambda-k)(\lambda-\theta_1)}{k(\theta_1+1)} \end{aligned}\right\} \text{— (i)}$$

If $\beta = \theta_D$

$$\left.\begin{aligned} a_0(W) &= -\frac{b_1}{\theta_D+1} - 1 \\ f_W(\lambda) &= \frac{(\lambda-k)(\lambda-\theta_D)}{k(\theta_D+1)}, \end{aligned}\right\} \text{— (ii)}$$

If $\beta \notin \{\theta_1, \theta_D\}$.

$\beta \in (-\infty, \theta_D) \cup (\theta_1, \infty)$

we have (iv)

Note using $(*)$

$a_0(W) \rightarrow \beta \rightarrow f_W \rightarrow m_W \rightarrow$ isomorphism class of W.

Note on LEMMA 29

In fact $\theta_1 > -1$ , $\theta_D < -1$ if $D \geq 2$

DEF. The complete graph $K_n$ has $n$ vertices and diameter $D = 1$ i.e., $xy \in E$ for $\forall$ vertices $x, y$.

$K_n$ is distance-regular with valency $k = n-1$

$a_1 = n-2$

$D = 1$. 2-distinct eigenvalues $\theta_0, \theta_1$.

Recall $\theta_0, \cdots, \theta_D$ are roots of

$p_{D+1}$ : $D+1$ st polynomial for the trivial module.

$$p_0 = 1$$
$$p_1 = \lambda$$
$$\begin{aligned} p_2 &= \lambda^2 - a_1\lambda - k \\ &= \lambda^2 - (n-2)\lambda - (n-1) \\ &= (\lambda - (n-1))(\lambda + 1) \end{aligned}$$

The roots are $\theta_0 = n-1 = k$

$\theta_1 = -1$.

LEMMA 30 Let $\Gamma = (X, E)$ be distance-regular of diameter $D \geq 1$ with distinct eigenvalues

$k = \theta_0 > \theta_1 > \cdots > \theta_D$

(i) $\theta_D \leq -1$ with equality iff $D = 1$.

(ii) $\theta_1 \geq -1$ with equality iff $D = 1$

Proof. (i) Suppose $\theta_D \geq -1$.

Then $I + A$ is positive semi-definite.

By Lemma 4. there exist vectors $\{v_x \mid x \in X\}$ in a Euclidean space s.t

$$\langle v_x, v_y \rangle = (I + A)_{xy} = \begin{cases} 1 & \text{if } x = y \text{ or } xy \in E \\ 0 & \text{otherwise} \end{cases}$$

$\forall xy \in E$

$$\langle v_x, v_y \rangle = \| v_x \| \| v_y \| = 1$$

Hence $v_x = v_y$.

and $v_x$ is independent of $x \in X$.

Thus $\langle v_x, v_y \rangle = 1 \quad \forall x, y \in X$.

We have $I + A = J$ (all 1's matrix)

$\therefore D = 1$.

(ii) Let $m$ be the trivial measure.

$$1 = \sum_{\theta \in \mathbb{R}} m(\theta) + \sum_{\theta \in \mathbb{R}} m(\theta)\theta = \sum_{\theta \in \mathbb{R}} m(\theta)(\theta + 1)$$

$$= m(k)(k+1) + \sum_{\theta \neq k} m(\theta)(\theta + 1)$$

$$\leq (k+1)|X|^{-1} \qquad (\because m(k) = |X|^{-1} \dim E_0 V = |X|^{-1})$$

So $k + 1 \geq |X|$ or $k = |X| - 1$.

$xy \in E \ \forall x, y \in X$ and $D = 1$

Note : Lemma does not require distance-regular assumption

Lecture 16 Wed. Feb. 24, 1993

Let $\Gamma = (X, E)$ denote any graph of diameter $D$.

DEF. For all integers $i$, the $i$-th incidence matrix $A_i \in \mathrm{Mat}_X(\mathbb{C})$ satisfies

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \qquad (x, y \in X)$$

Observe
$A_0 = I$ (identity)
$A_1 = A$ (adjacency matrix)
$A_0 + A_1 + \cdots + A_D = J$ (all 1's matrix)

In general $A_i \notin$ Bose Mesner algebra

LEMMA 31. Assume $\Gamma = (X, E)$ is distance-regular with diameter $D \geq 1$, intersection numbers $c_i, a_i, b_i$.

(i) $AA_i = c_{i+1} A_{i+1} + a_i A_i + b_{i-1} A_{i-1}$
$(0 \leq i \leq D, \quad A_{-1} = A_{D+1} = 0)$

(ii) $$A_i = \frac{p_i(A)}{c_1 c_2 \cdots c_i} \qquad (0 \leq i \leq D).$$

where $p_0, \ldots, p_D$ are the polynomials for the trivial module from Lemma 15 (Lec 9-1)

(iii) $A_0, A_1, \ldots A_D$ form a basis for Bose-Mesner algebra $M$.

(iv) $\forall$ distances $h, i, j$ $(0 \leq i, j, h \leq D)$, and $\forall x, y \in X$ with $\partial(x, y) = h$, the constant

$$p^h_{ij} = |\{ z \in X \mid \partial(x, z) = i, \ \partial(y, z) = j \}|$$

depends only on $h, i, j$ not on $x, y$.

(v) $E_0 = \dfrac{1}{|X|} J$.

Proof

(i) Pick $x \in X$. Apply each side to $\hat{x}$.

$$A A_i \hat{x} \overset{?}{=} c_{i+1} A_{i+1} \hat{x} + a_i A_i \hat{x} + b_{i-1} A_{i-1} \hat{x}.$$

LHS :

$$= A \Big( \sum_{y \in X\ \partial(x,y)=i} \hat{y} \Big)$$

$$= c_{i+1} \Big( \underbrace{\sum_{z \in X,\ \partial(x,z)=i+1} \hat{z}}_{A_{i+1}\hat{x}} \Big) + a_i \Big( \underbrace{\sum_{z \in X,\ \partial(x,z)=i} \hat{z}}_{A_i \hat{x}} \Big) + b_{i-1} \Big( \underbrace{\sum_{z \in X,\ \partial(x,z)=i-1} \hat{z}}_{A_{i-1}\hat{x}} \Big)$$

$= \text{RHS}.$

(ii) Recall (Lemma 15 Lec 9-1)

$$A p_i(A) = p_{i+1}(A) + a_i p_i(A) + b_{i-1} c_i p_{i-1}(A) \quad (0 \le i \le D)$$

Dividing by $c_1 \cdots c_i$, we have

$$A \Big( \frac{p_i(A)}{c_1 c_2 \cdots c_i} \Big) = c_{i+1} \Big( \frac{p_{i+1}(A)}{c_1 c_2 \cdots c_{i+1}} \Big) + a_i \Big( \frac{p_i(A)}{c_1 c_2 \cdots c_i} \Big) + b_{i-1} \Big( \frac{p_{i-1}(A)}{c_1 c_2 \cdots c_{i-1}} \Big)$$

So $A_i$, $\dfrac{p_i(A)}{c_1 c_2 \cdots c_i}$ satisfy the same recurence.

Also boundary condition

$$A_0 = p_0(A) = I.$$

Hence $$A_i = \frac{p_i(A)}{c_1 c_2 \cdots c_i} \quad (0 \le i \le D).$$

(iii) Since $E_0, E_1, \cdots, E_D$ form a basis for $M$

$\dim M = D+1$.

Observe : $A_0, A_1, \cdots, A_D \in M$ by (ii)

$A_0, A_1, \cdots A_D$ lin. indep. since $p_0, \cdots, p_D$ lin indep.

Thus $A_0, A_1, \dots, A_D$ form a basis for $M$.

(iv). $A_0, A_1, \ldots, A_D$ form a basis for an algebra $M$,

$$A_i A_j = \sum_{l=0}^{D} p_{ij}^{l} A_l \qquad \text{for some } p_{ij}^{l} \in \mathbb{C}. \quad \text{—}(\ast)$$

Fix $h$ $(0 \le h \le D)$. Pick $x, y \in X$ with $\partial(x,y) = h$

Compute $x, y$ entry in $(\ast)$

$$(A_i A_j)_{xy} = \sum_{z \in X} (A_i)_{xz} (A_j)_{zy}$$

$$= \sum_{z \in X,\, \partial(x,z) = i,\, \partial(y,z) = j} 1 \cdot 1$$

$$= \left| \{ z \in X \mid \partial(x,z) = i,\ \partial(y,z) = j \} \right|$$

On the other hand

$$\left( \sum_{l=0}^{D} p_{ij}^{l} A_l \right)_{xy} = p_{ij}^{h} (A_h)_{xy}$$

$$= p_{ij}^{h}$$

(v) $\frac{1}{|X|} J$ is the orthogonal projection onto

$$\mathrm{Span}(\delta) = E_0 V$$

Hence $\frac{1}{|X|} J = E_0$.

**Theorem 32.** Let $\Gamma = (X, E)$ be distance regular with diameter $D \geq 2$ intersection numbers $c_i, a_i, b_i$. Pick $x \in X$. Let $W$ be a thin irreducible $T(x)$-module with endpoint $r = 1$ and diameter $d$. $(d = D-2$ or $D-1)$

Set $\gamma_0 = a_0(W) + 1$.

(i) The scalars

$$\gamma_i := \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_1(W) x_2(W) \cdots x_i(W)} \qquad (0 \leq i \leq d) \quad -(1)$$

$a_i(W)$, $x_i(W)$ are algebraic integers in $\mathbb{Q}[\gamma_0]$

In particular, if $\gamma_0 \in \mathbb{Q}$, then $\gamma_i$, $a_i(W)$, $x_i(W)$ are integers $\forall i$.

(ii) The numbers $\gamma_i$, $a_i(W)$, $x_i(W)$ can all be determined from $\gamma_0$ and the intersection numbers of $\Gamma$ in order

$x_1(W)$, $\gamma_1$, $a_1(W)$, $x_2(W)$, $\gamma_2$, $a_2(W)$ ..

using (i)

$$x_i(W) = c_i b_i + \gamma_{i-1}(a_i + c_i - c_{i+1} - a_{i-1}(W)) \quad -(2)$$
$$(1 \leq i \leq D-1)$$

and

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \quad -(3)$$
$$(1 \leq i \leq d)$$

(Note: $p_i = p_1^W + \gamma_{i-1} p_{i-1}^W - c_i(p_{i-1}^W + \gamma_{i-2} p_{i-2}^W)$

$(\gamma_{-1} = -\gamma_{-2} = 0, \quad 0 \leq i \leq d+1)$.

Proof.

Set $\tilde{A}_i = A_0 + A_1 + \cdots + A_i \qquad (0 \le i \le D)$

<u>Claim 1</u> $\quad A\tilde{A}_i = c_{i+1}\tilde{A}_{i+1} + (a_i - c_{i+1} + c_i)\tilde{A}_i + b_i \tilde{A}_{i-1} \quad (0 \le i \le D-1)$

<u>Pf of Claim 1</u> $\quad$ LHS :

$$= \sum_{j=0}^{i} A A_j$$

$$= \sum_{j=0}^{i} (c_{j+1}A_{j+1} + a_j A_j + b_{j-1}A_{j-1})$$

$$= \sum_{j=0}^{i-1} A_j(c_j + a_j + b_j) + A_i(c_i + a_i) + A_{i+1}c_{i+1}$$

$$= k(A_0 + \cdots + A_{i-1}) + (a_i + c_i)A_i + c_{i+1}A_{i+1}$$

RHS :

$$= c_{i+1}(A_0 + A_1 + \cdots + A_{i-1} + A_i + A_{i+1})$$
$$+ (a_i - c_{i+1} + c_i)(A_0 + A_1 + \cdots + A_{i-1} + A_i)$$
$$+ b_i (A_0 + A_1 + \cdots + A_{i-1})$$
$$= k(A_0 + \cdots + A_{i-1}) + A_i(a_i + c_i) + A_{i+1}c_{i+1}.$$

This proves Claim 1.

Now pick $\quad 0 \neq w \in E_1^*(x)W$

$$w = \sum_{\substack{z \in X \\ \partial(x,z)=1}} \alpha_z \hat{z}.$$

Pick $y$ where $\alpha_y \neq 0$.

$\forall i \quad (0 \le i \le D) \quad$ define

$$B_i = \tilde{A}_i(\hat{x} - \hat{y})$$

$$= \sum_{\substack{z \in X \\ \partial(x,z) \le i}} \hat{z} - \sum_{\substack{z \in X \\ \partial(y,z) \le i}} \hat{z}$$

$$= \sum_{\substack{z \in X \\ \partial(x,z)=i \\ \partial(y,z)=i+1}} \hat{z} - \sum_{\substack{z \in X \\ \partial(x,z)=i+1,\ \partial(y,z)=i}} \hat{z}$$

Note $\quad B_D = 0. \qquad B_0 = \hat{x} - \hat{y}$

$$\langle B_0, w \rangle = -\alpha_y \neq 0$$

From claim 1

$$AB_i = c_{i+1} B_{i+1} + (a_i - c_{i+1} + c_i) B_i + b_i B_{i-1} \qquad (0 \le i \le D)$$

$$B_{-1} = 0$$

Let $p_0^W, \dots, p_d^W$ denote polynomials for $W$ from Lemma 15 (Lec 9-1)

So

$$w_i = p_i^W(A) w \in E^*_{t+i}(x) W \qquad (0 \le i \le d)$$

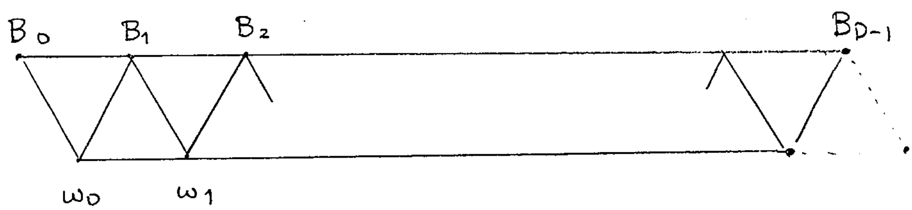
<u>Claim 2</u> $\quad \langle w_i, B_j \rangle = 0 \quad$ if $j \notin \{i, i+1\}$

$(0 \le i \le d, \quad 0 \le j \le D)$.

<u>pf. of Claim 2</u>

$$w_i \in E^*_{t+i}(x) W$$

$$B_j \in E^*_j(x) W + E^*_{j+1}(x) W$$




vertical lines indicate possible non-orthogonality.

- Compute $\quad \langle Aw_i, B_j \rangle = \langle w_i, AB_j \rangle \quad$ —(*)

$(0 \le i \le d, \quad 0 \le j \le D-1)$

LHS

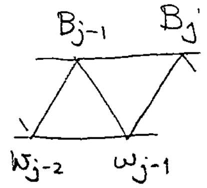
$$= \langle w_{i+1}, B_j \rangle + a_i(W) \langle w_i, B_j \rangle + x_i(W) \langle w_{i-1}, B_j \rangle$$

RHS

$$= b_j \langle w_i, B_{j-1} \rangle + (a_j - c_{j+1} + c_j) \langle w_i, B_j \rangle + c_{j+1} \langle w_i, B_{j+1} \rangle$$

Evaluate for $i = j-2,\ j-1,\ j,\ j+1.$

Set $i = j-2$:




$(*)$ becomes

$$\langle w_{j-1}, B_j \rangle = b_j \langle w_{j-2}, B_{j-1} \rangle \qquad (2 \le j \le D-1)$$

By induction

$$\langle w_{j-1}, B_j \rangle = b_2 b_3 \cdots b_j \langle w_0, B_1 \rangle \qquad (1 \le j \le D-1)$$

Define $\gamma_0 = \dfrac{\langle w_0, B_1 \rangle}{\langle w_0, B_0 \rangle}$ (we will show $\gamma_0 = 1 + a_0(W)$)

Then

$$\langle w_{j-1}, B_j \rangle = b_2 b_3 \cdots b_j \gamma_0 \langle w_0, B_0 \rangle \qquad \text{—— (4).}$$

Set $i = j+1$:

$(*)$ becomes

$$x_{j+1}(W) \langle w_j, B_j \rangle = c_{j+1} \langle w_{j+1}, B_{j+1} \rangle \qquad (0 \le j \le d)$$

Hence

$$\langle w_j, B_j \rangle = \frac{x_1(W) \cdots x_j(W)}{c_1 c_2 \cdots c_j} \langle w_0, B_0 \rangle \qquad (0 \le j \le d) \text{ -(5)}$$

Set $i = j-1$:

$(*)$ becomes

$$\langle w_j, B_j \rangle + a_{j-1}(W) \langle w_{j-1}, B_j \rangle$$
$$= (a_j - c_{j+1} + c_j) \langle w_{j-1}, B_j \rangle + b_j \langle w_{j-1}, B_{j-1} \rangle$$

Evaluate this using (4)(5)

we have $\qquad (\langle w_0, B_0 \rangle \neq 0)$

$$\langle w_j, B_j\rangle + a_{j-1}(W)\langle w_{j-1}, B_j\rangle$$
$$= (a_j - c_{j+1} + c_j)\langle w_{j-1}, B_j\rangle + b_j\langle w_{j-1}, B_{j-1}\rangle$$

$$\frac{x_1(W)\cdots x_j(W)}{c_1\cdots c_j} + (a_{j-1}(W) - a_j + c_{j+1} - c_j)\, b_2\cdots b_j\gamma_0 = b_j\frac{x_1(W)\cdots x_{j-1}(W)}{c_1\cdots c_{j-1}}$$

$$\left(\ \gamma_i := \frac{c_2c_3\cdots c_{i+1}\, b_2b_3\cdots b_{i+1}\gamma_0}{x_1(W)x_2(W)\cdots x_i(W)}\ \right)$$

$$\frac{x_j(W)}{c_j} = b_j + \frac{c_1\cdots c_{j-1}b_2\cdots b_j\gamma_0}{x_1(W)\cdots x_{j-1}(W)}(a_j + c_j - c_{j+1} - a_{j-1}(W))$$

$$\therefore\ x_j(W) = c_jb_j + \gamma_{j-1}(a_j + c_j - c_{j+1} - a_{j-1}(W))$$

This proves (2)

Set $i=j$

(*) becomes

$$a_j(W)\langle w_j, B_j\rangle + x_j(W)\langle w_{j-1}, B_j\rangle$$
$$= (a_j - c_{j+1} + c_j)\langle w_j, B_j\rangle + c_{j+1}\langle w_j, B_{j+1}\rangle$$

$$\Big(a_j(W) - (a_j - c_{j+1} + c_j)\Big)\frac{x_1(W)\cdots x_j(W)}{c_1\cdots c_j}$$
$$+ x_j(W)\, b_2\cdots b_j\gamma_0 - c_{j+1}b_2\cdots b_{j+1}\gamma_0 = 0$$

$$a_j(W) - (a_j - c_{j+1} + c_j) + \frac{c_1\cdots c_j b_2\cdots b_j\gamma_0}{x_1(W)\cdots x_{j-1}(W)} - \frac{c_1\cdots c_jc_{j+1}b_2\cdots b_{j+1}\gamma_0}{x_1(W)\cdots x_j(W)} = 0$$

or

$$a_j(W) = a_j + c_j - c_{j+1} - \gamma_{j-1} + \gamma_j$$

This proves (3).

Also setting $i=j=0$ we find

$$a_0(W)\langle w_0, B_0\rangle = (a_0 - c_1 + c_0)\langle w_0, B_0\rangle + c_1\langle w_0, B_1\rangle$$
$$= -\langle w_0, B_0\rangle + \gamma_0\langle w_0, B_0\rangle$$

Hence

$$\gamma_0 = 1 + a_0(W)$$

$a_i(W)$, $x_i(W)$ are algebraic integers since they are eigenvalues of matrices with integer entries, namely

$$E^*_{i+1}(x)\,A\,E^*_{i+1}(x) \quad \text{and}$$
$$E^*_i(x)\,A\,E^*_{i+1}(x)\,A\,E^*_i(x).$$

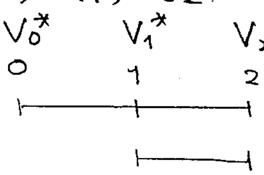
Also $\gamma_0 = 1 + a_0(W)$ is an algebraic integer.
$\gamma_i - \gamma_{i-1}$ is an algebraic integer by (3).
Hence $\gamma_i$ is an algebraic integer by Induction

This completes the proof of Thm 32.

<u>Example</u> $D=2$ $\Leftrightarrow$ strongly regular

Free parameters

$$k,\ a_1,\ c_2.$$




trivial module

W :

Matrix representation $A|W$ is

$$\begin{pmatrix} a_0(W) & x_1(W) \\ 1 & a_1(W) \end{pmatrix}$$

$a_0(W)$ : free

$$\begin{aligned} x_1(W) &= c_1 b_1 + (a_0(W)+1)(a_1 + c_1 - c_2 - a_0(W)) \\ &= k \cancel{- a_1 - 1} + a_1 a_0(W) \cancel{+ a_0(W)} - c_2 a_0(W) - a_0(W)^2 \cancel{- a_0(W)} \cancel{+ a_1 + 1} - c_2 \\ &= a_1 a_0(W) - c_2 a_0(W) + k - c_2 - a_0(W)^2 \end{aligned}$$

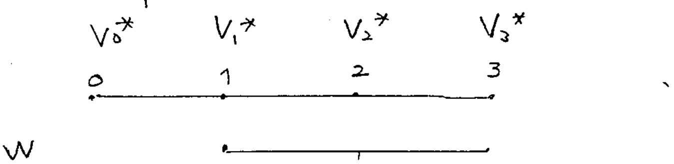
$$\gamma_1 = 0$$

$$\begin{aligned} a_1(W) &= -(a_0(W)+1) + a_1 + c_1 - c_2 \\ &= -a_0(W) + a_1 - c_2 \end{aligned}$$

Then the matrix has eigenvalues $\theta$, $\theta_1$
There is one feasible condition : $a_0(W)$ is alg. integer

## Example D=3

Free parameters $c_2$, $c_3$, $k$, $a_1$, $a_2$




Matrix rep.

$$A|W = \begin{pmatrix} a_0(W) & x_1(W) & 0 \\ 1 & a_1(W) & x_2(W) \\ 0 & 1 & a_2(W) \end{pmatrix}$$

$$a_0(W) \quad : \quad \text{free} \qquad (= \gamma_0 - 1)$$
$$x_1(W) = k - 1 - a_1 + \gamma_0(a_1 + 1 - c_2 - a_0(W))$$
$$= \gamma_0(a_1 - c_2 - a_0(W)) + k - a_1 + a_0(W)$$

Set $\quad \gamma_1(W) = \dfrac{c_2 b_2 \gamma_0}{x_1(W)}$

$$a_1(W) = \gamma_1 - \gamma_0 + a_1 + 1 - c_2$$
$$x_2(W) = \gamma_1(a_2 - c_3 - a_1(W)) + c_2(\gamma_0 + b_1 - a_2 + a_1(W))$$
$$a_2(W) = -\gamma_1 + a_2 + c_2 - c_3$$

Then matrix has eigenvalues $\theta_1$, $\theta_2$, $\theta_3$
There are 2 feasibility conditions ;
$\gamma_0$, $\gamma_1$ are alg integers.
for arbitrary D. there are D−1 feasibility conditions
$\gamma_0, \gamma_1, \ldots, \gamma_{D-2}$ are alg. integers

Lemma 33. With the notation of Theorem 32

Suppose $f_W = \dfrac{k-\lambda}{k}$ (so $a_0(W) = -1$)

Then

$$a_i(W) = a_i + c_i - c_{i+1} \qquad (0 \le i \le D-1)$$
$$x_i(W) = b_i c_i \qquad (1 \le i \le D-1)$$
$$\gamma_i(W) = 0 \qquad (0 \le i \le D-1)$$

Proof

$$\gamma_0 = a_0(W) + 1$$

So $\gamma_i = 0$

No.

Date

# Lecture 17 Mon. March 1, 1993

Review

$\Gamma = (X, E)$ : distance-regular of diameter $D \geq 2$

Pick $x \in X$

Let $W$ be a thin irreducible $T(x)$-module with endpoint $r=1$, diameter $d = D-1$ or $D-2$

$\gamma_0 = a_0(W) + 1$

Show $$\gamma_i = \frac{c_2 c_3 \cdots c_{i+1} \, b_2 b_3 \cdots b_{i+1} \, \gamma_0}{x_1(W) \cdots x_i(W)},$$

$a_i(W)$, $x_i(W)$ are all algebraic integers in $\mathbb{Q}[\gamma_0]$, where

$$x_i(W) = c_i b_i + \gamma_{i-1}(a_i + c_i - c_{i+1} - a_{i-1}(W)) \quad (1 \leq i \leq d)$$

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \quad (1 \leq i \leq d)$$

Certainly $x_i(W)$, $\gamma_i$, $a_i(W) \in \mathbb{Q}[\gamma_0]$ by the above lines and so on

$$\gamma_0 \to a_0(W) \to x_1(W) \to \gamma_1 \to a_1(W) \to x_1(W) \to$$

Recall some $B \in \mathrm{Mat}_n(\mathbb{C})$ is <u>integral</u> whenever $B \in \mathrm{Mat}_n(\mathbb{Z})$

In this case, the characteristic polynomial

$$\det(\lambda I - B) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$$

some $\alpha_0, \cdots, \alpha_{n-1} \in \mathbb{Z}$.

Hence eigenvalues of $B$ are <u>algebraic</u> <u>integers</u>

But $a_i(W)$ is an eigenvalue of an integral matrix $B = E^*_{i+1}(x) A E^*_{i+1}(x)$.

Hence $a_i(W)$ is an algebraic integer

Also $x_i(W)$ is an eigenvalue of an integral matrix

$$B = E_i^*(x) A E_{i+1}^*(x) A E_i^*(x).$$

So $x_i(W)$ is an algebraic integer.

$\gamma_i - \gamma_{i-1} = a_i(W) - a_i - c_i + c_{i+1}$

is an algebraic integer

Since $\gamma_0 = a_0(W) + 1$ is an algebraic integer,

we find $\gamma_i$ is an algebraic integer for $\forall i$.

DEF. A (commutative) association scheme is a configuration $Y = (X, \{R_i\}_{0 \le i \le D})$, where

$X$ is a finite nonempty set (of vertices)

$R_0, R_1, \cdots, R_D$ are non empty subsets of $X \times X$ s.t.

(i) $R_0 = \{(x,x) \mid x \in X\}$

(ii) $R_0 \cup \cdots \cup R_D = X \times X$ (disj. union)

(iii) $\forall i$, $R_i^t = \{(y,x) \mid xy \in R\}$
$= R_{i'}$ some $i' \in \{0, 1, \cdots, D\}$

(iv) $\forall h, i, j$ $(0 \le h, i, j \le D)$ $\forall x, y$ s.t. $(x,y) \in R_h$
$p_{ij}^h = |\{z \in X \mid (x,z) \in R_i, (z,y) \in R_j\}|$
depends only on $h, i, j$ not on $x, y$.

(v) $p_{ij}^h = p_{ji}^h$ $\forall h, i, j$.

If $i' = i$ $\forall i$, we say $Y$ is symmetric.
$D$ = class of scheme
$R_i$ = $i$-th relation of $Y$.
Say vertices $x, y \in X$ are $i$-related or 'at distance $i$' whenever $(x,y) \in R_i$.

Assume scheme $\leftrightarrow$ commutative association scheme.

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be an association scheme

DEF The $i$-th association matrix $A_i \in \mathrm{Mat}_X(\mathbb{C})$

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{if } (x,y) \in R_i \end{cases} \qquad \begin{matrix} (x, y \in X) \\ (0 \le i \le D) \end{matrix}$$

Then

(i') $A_0 = I$

(ii') $A_0 + A_1 + \cdots + A_D = J$ (= all 1's matrix)

(iii') $A_i^t = A_{i'}$ $(0 \le i \le D)$

(iv') $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h$ $(0 \le i, j \le D)$

(v') $A_i A_j = A_j A_i$

$M := \mathrm{Span}_{\mathbb{C}} \{A_0, \cdots, A_D\}$ (Bose-Mesner algebra of $Y$)
is a commutative $\mathbb{C}$-algebra of dimension $D$

Observe:

$Y$ is symmetric $\leftrightarrow$ $A_i^t = A_i$ $\forall i$
$\leftrightarrow$ $M$: symmetric

Example 1. Let $\Gamma = (X, E)$ be distance-regular of diameter $D$

Set

$$R_i = \{(x, y) \mid \partial(x,y) = i\} \qquad (0 \le i \le D)$$

then

$Y = (X, \{R_i\}_{0 \le i \le D})$ is a symmetric scheme

$i$-th association matrix
$= i$-th distance matrix $\forall i$

Example 2  Suppose a group $G$ acts transitively on a set $X$. Assume $G$ is generously transitive, i.e.,
$\forall x, y \in X \quad \exists g \in G$ s.t $\quad gx = y \quad gy = x$
$G$ acts on $X \times X$ by rule
$g(x, y) = (gx, gy) \quad \forall g \in G, \quad \forall x, y \in X.$

Let $R_0, \cdots, R_D$ denote orbits of $G$ on $X \times X$

Observe $\quad R_i^t = R_i \quad \forall i \quad$ by generously transitivity.
$Y = \{X, \{R_i\}_{0 \le i \le D}\}$
is a symmetric scheme

Exercise  In example 2, Bose Mesner algebra
$M = \{ B \in \mathrm{Mat}_X(\mathbb{C}) \mid Bg = gB \quad \forall g \in G \}$
$=$ commuting algebra of $G$ on $X$.
(Here we view each $g \in G$ as a permutation matrix $\in \mathrm{Mat}_X(\mathbb{C})$
satisfying $\quad g\hat{x} = \widehat{gx} \quad \forall x \in G$ )

Example 3  Let $G$ be any finite group
$G$ acts on $X = G$ by conjugation
$G \times X \to X \qquad (g, x) \to gxg^{-1}$
let $C_0, \cdots, C_D$ denote orbits (i.e., conjugacy classes)
($C_0 = \{1_G\}$)
Define $\quad R_i = \{xy \mid x, y \in X, \; x^{-1}y \in C_i\} \quad (0 \le i \le D)$

Claim $\quad Y = (X, \{R_i\}_{0 \le i \le D})$
is commutative scheme (not symmetric in general)

(i) $R_0 = \{xx \mid x \in X\} \leftarrow C_0 = \{1_G\}$

(ii) $R_0, \cdots, R_D$ partition $X \times X$

since $C_0, \cdots, C_D$ partition $X = G$

(iii) $R_i^t = R_{i'}$, where $C_{i'} = \{g^{-1} \mid g \in C_i\}$

(iv) Set $H = G \oplus G$ direct sum.

$H$ acts on $X = G$ :

$\forall h = (g, gz) \quad \forall x \in X.$

$h(x) = gx(gz)^{-1} = gxz^{-1}g^{-1}$

$R_i = \{(x, y) \mid x^{-1}y \in C_i\}$

$h_i \in C_i \quad x^{-1}y = gh_ig^{-1}$

$(x, y) = (x, \ xgh_ig^{-1})$

$= (xgg^{-1}, \ xgh_ig^{-1})$

$= (xg, g)(1, h_i)$

So

$R_0, \cdots, R_D$ orbits of $H$ on $X \times X$.

(v) $p_{ij}^h = p_{ji}^h$ ?

Fix $i, j, h$

Fix $x, y \in X$ with $(x, y) \in R_h$

Set

$S = \{z \in X \mid (x, z) \in R_i \quad (z, y) \in R_j\}$

$T = \{z \in X \mid (x, z) \in R_j \quad (z, y) \in R_i\}$

Show $|S| = |T|$

$\forall z \in S$ set $\hat{z} = xz^{-1}y$.

Observe $\hat{z} \in T$

$x^{-1}z \in C_i \qquad x^{-1}\hat{z} = x^{-1}xz^{-1}y \in C_j$

$z^{-1}y \in C_j \qquad \hat{z}^{-1}y = y^{-1}zx^{-1}y = y^{-1}x(x^{-1}z)x^{-1}y \in C_i$

Observe $S \to T$ is 1-1 onto

$z \to z^{-1}$

# Lecture 18 Wed March 3, 1993

LEMMA 33 Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote symmetric scheme with associated matrices $A_0, \cdots, A_D$. Then the following are equivalent

(i) The graph $\Gamma = (X, R_1)$ is distance-regular (and $R_0, \cdots, R_D$ are labelled so that $R_i = \{xy \mid \partial(x,y) = i\}$.)

(ii) $\exists f_i \in \mathbb{C}[\lambda]$ $\deg f_i = i$ s.t. $f_i(A_1) = A_i$ $(0 \le i \le D)$

(iii) $p^h_{ij} \begin{cases} = 0 & \text{if one of } h, i, j > \text{sum of other 2} \\ \neq 0 & \text{if one of } h, i, j = \text{sum of other 2} \end{cases}$

Proof

(i) $\Rightarrow$ (ii) Lemma 31

(ii) $\Rightarrow$ (iii) Define $k_i \equiv p^0_{ii}$ $\quad (x, z) \in R_i$

$= |\{z \mid z \in X, \partial(x,z) = i\}|$

for any $x \in X$.

Then $k_i \neq 0$ $(0 \le i \le D)$ $\quad k_0 = 1$

(by symmetricity $(x, y) \in R_i \leftrightarrow (y, x) \in R_i$)

Claim $k_h p^h_{ij} = k_i p^i_{hj} = k_j p^j_{ih}$

$= |X|^{-1} |\{xyz \in X^3 \mid \partial(x,y) = h \ \partial(x,z) = i, \partial(y,z) = j\}|$

(pf) # of $xyz \in X^3$ $\partial(x,y) = h$ $\partial(x,z) = i$ $\partial(y,z) = j$

$= |X| k_h p^h_{ij} = |X| k_i p^i_{hj} = k_j p^j_{ih}$

In particular, $p^h_{ij} = 0 \leftrightarrow p^i_{hj} = 0 \leftrightarrow p^j_{ih} = 0$

Hence it suffices to show

$p^h_{ij} \begin{matrix} = 0 & \text{if } h > i+j \\ \neq 0 & \text{if } h = i+j \end{matrix}$

Fix $i, j$. WLOG $i+j \le D$ else trivial

$$f_i(A) f_j(A) = A_i A_j = \sum_{l=0}^{D} p_{ij}^{l} A_l = \sum_{l=0}^{D} p_{ij}^{l} f_l(A)$$

$$\begin{aligned} i+j &= \deg \text{LHS} \\ &= \deg \text{RHS} \\ &= \max \{ l \mid p_{ij}^{l} \ne 0 \} \end{aligned}$$

(iii) $\Rightarrow$ (i) Let $A = A_1$ and consider a graph $\Gamma$ with adjacency matrix $A$.

$$\begin{aligned} A A_j &= \sum_h p_{1j}^{h} A_h \\ &= \underset{\ne 0}{p_{1j}^{j+1}} A_{j+1} + p_{1j}^{j} A_j + \underset{\ne 0}{p_{1j}^{j-1}} A_{j-1} \end{aligned}$$

Fix $x \in X$.

Set $R_i(x) = \{ y \mid (x, y) \in R_i \}$.

Then each $y \in R_i(x)$ is adjacent (in $\Gamma$) to exactly

$p_{1\,i+1}^{i} \ne 0$ vertices in $R_{i+1}(x)$

$p_{1i}^{i}$ vertices $"$ $R_i(x)$

$p_{1\,i-1}^{i} \ne 0$ vertices $"$ $R_{i-1}(x)$.

Hence by induction

$R_i(x) = \{ y \mid \partial(x, y) = i$ in $\Gamma \}$ $(0 \le i \le D)$

and $\Gamma$ is distance regular

Lecture 19 Fri. March 5, 1993

LEMMA 34 Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme with Bose-Mesner algebra $M$. Then there exists a basis $E_0, \dots, E_D$ for $M$ s.t.

(i) $E_0 = |X|^{-1} J$

(ii) $E_i E_j = E_j E_i = \delta_{ij} E_i \qquad (0 \le i, j \le D)$

(iii) $E_0 + E_1 + \cdots + E_D = I$

(iv) $E_i^t = \overline{E}_i$
$= E_{\hat{i}}$ some $\hat{i} \in \{0, 1, \dots, D\}$.

Proof. $M$ acts on Hermitean space $V = \mathbb{C}^n$ $(n = |X|)$.
If $W$ is an $M$-module, so is $W^\perp$.
Each irreducible $M$-module is 1 dimensional by commutativity of $M$.
So $V$ is an orthogonal direct sum of 1 dimensional $M$-modules.
Let $v_1, \dots, v_n$ be an orthonormal basis for $V$ consisting of eigenvectors for $\forall m \in M$.

Set $P \in \mathrm{Mat}_X(\mathbb{C})$ so that the $i$-th column of $P = v_i$.
So

$$\overline{P}^t P = I = P \overline{P}^t = \overline{P} P^t$$

and $P$ is unitary.
Also for $\forall m \in M$,

$$P^{-1} m P = \text{diagonal} = \mathrm{diag}(\theta_1(m), \dots, \theta_n(m))$$

for some functions

$$\theta_i : M \longrightarrow \mathbb{C}.$$

Observe: each $\theta = \theta_i$ is a <u>character</u> of $M$, i.e, $\theta : M \to \mathbb{C}$ is a $\mathbb{C}$-alg. homomorphism

No. Lec 19−2

Date 1995 · 2 · 12

Observe : the $\theta_1, \cdots, \theta_n$ are not all distinct.

Let $\sigma_0, \cdots, \sigma_r$ denote distinct elements of $\theta_1, \cdots, \theta_n$.

Say $\sigma_i$ appear $m_i$ times.

WLOG

$$P^{-1} m P = \left(\begin{array}{ccc|ccc|c|ccc} \sigma_0(m) & & & & & & & & & \\ & \ddots & & & & & & & 0 & \\ & & \sigma_0(m) & & & & & & & \\ \hline & & & \sigma_1(m) & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \sigma_1(m) & & & & \\ \hline & & & & & & \ddots & & & \\ \hline & 0 & & & & & & \sigma_r(m) & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \sigma_r(m) \end{array}\right)$$

Set

$$\begin{array}{c} \text{$i$-th block} \\ E_i = P \left(\begin{array}{c|ccc|c} & & & & \\ \hline & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \hline & & & & \end{array}\right) P^{-1} \end{array}$$

Then

$$E_i E_j = \delta_{ij} E_i \quad (0 \le i, j \le r)$$

$$E_0 + E_1 + \cdots + E_r = I$$

$\forall m \in M$.

$$m = \sum_{i=0}^{r} \sigma_i(m) E_i$$

$$\in \mathrm{Span}(E_0, \cdots, E_r)$$

So

$$M \subseteq \mathrm{Span}(E_0, \cdots, E_r)$$

Since

$E_0, \cdots, E_r$ are linearly independent,

$r \ge D$.

Show $E_i \in M$.

Claim 1. $\forall$ distinct $i, j$ $(0 \le i, j \le D)$
there $\exists$ $m \in M$ $\sigma_i(m) \neq 0$, $\sigma_j(m) = 0$.

pf of claim 1

$\sigma_i \neq \sigma_j$ implies
$\exists$ $m' \in M$ $\sigma_i(m') \neq \sigma_j(m')$.

Set $m = m' - \sigma_j(m') I$

Then

$$\sigma_j(m) = \sigma_j(m') - \sigma_j(m') = 0$$
$$\sigma_i(m) = \sigma_i(m') - \sigma_j(m') \neq 0.$$

Claim 2 $E_i \in M$ $(0 \le i \le D)$.

pf of claim 2

Fix $x$

For $\forall j \neq i$, $\exists$ $m_j \in M$ s.t.
$\sigma_i(m_j) \neq 0$, $\sigma_j(m_j) = 0$ $i \neq j$.

Observe

$$s = \sigma_i\Big(\prod_{l \neq i} m_l\Big)$$
$$\neq 0$$

Set

$$m^* := \Big(\prod_{l \neq i} m_i\Big) s^{-1}.$$

Observe

$$\sigma_i(m^*) = 1$$
$$\sigma_j(m^*) = 0 \qquad \forall j \neq i \quad (0 \le j \le D).$$

So

$$P^{-1} m^* P = \begin{array}{c} \begin{array}{ccc} \phantom{0} & \overbrace{\phantom{000}}^{i\text{-th block}} & \phantom{0} \end{array} \\ \left(\begin{array}{c|c|c} & & \\ \hline & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \\ \hline & & \end{array}\right) \end{array}$$

We have $E_i = m^*$
$\in M$.

Now $r = D$, $M = \mathrm{Span}(E_0, \cdots, E_D)$ and
$E_0, \cdots, E_D$ is a basis for $M$

Observe

$$P^{-1} E_i P = \begin{pmatrix} & \vdots & \\ \cdots & I & \cdots \\ & \vdots & \end{pmatrix}$$

implies

$$\overline{P}^t \overline{E}_i^t \overline{P^{-1}}^t = \begin{pmatrix} & \vdots & \\ \cdots & I & \cdots \\ & \vdots & \end{pmatrix}^t = P^{-1} E_i P$$

$$\| $$

$$P^{-1} \overline{E}_i^t P$$

Hence

$$\overline{E}_i^t = E_i$$

$E_0^t, \cdots, E_D^t$ are non-zero matrices satisfying

$$E_i^t E_j^t = \delta_{ij} E_i^t$$

$$E_0^t + E_1^t + \cdots + E_D^t = I$$

Each $E_i^t$ is a linear combination of $E_0, \cdots, E_D$
with coefficients that are 0 or 1
and for no 2 $E_i^t$s are coefficients of any $E_j$ both 1's
So

$E_0^t, \cdots, E_D^t$ is a permutation of $E_0, \cdots, E_D$

Observe $J = A_0 + \cdots + A_D \in M$
$|X|^{-1} J$ is an idempotent of rank 1.
So

WLOG

$$E_0 = |X|^{-1} J.$$

Define entrywise product $\circ$ on $\mathrm{Mat}_X(\mathbb{C})$.

$$A_i \circ A_j = \delta_{ij} A_i$$

So $M$ is closed under $\circ$

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^h E_h.$$

$q_{ij}^h$ is called Krein parameters of $Y$.

$q_{ij}^h \in \mathbb{R}$ ?

$$\overline{(E_i \circ E_j)}^t = |X|^{-1} \sum_h \overline{q_{ij}^h}\, \overline{E_h}^t = \frac{1}{|X|} \sum_h \overline{q_{ij}^h} E_h$$

$$= E_i \circ E_j$$

$$= \frac{1}{|X|} \sum_h q_{ij}^h E_h$$

Hence $q_{ij}^h = \overline{q_{ij}^h}$

Observe

$A_0, \cdots, A_D$, $E_0, \cdots, E_D$ are bases for $M$.

$\exists\, p_i(j)$ $q_i(j) \in \mathbb{C}$ s.t.

$$A_i = \sum_{j=0}^{D} p_i(j) E_j$$

$$E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j.$$

Taking transpose & conjugate we find

$$\overline{p_i(j)} = p_{i'}(j)$$

$$= p_i(\hat{j}) \qquad (0 \le i, j \le D)$$

$$\overline{q_i(j)} = q_{\hat{i}}(j)$$

$$= q_i(j') \qquad (0 \le i, j \le D)$$

Fix $x \in X$

Define $E_i^* \equiv E_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$
to be a diagonal matrix s.t.

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{if } (x,y) \notin R_i \end{cases} \qquad \begin{matrix} (0 \le i \le D) \\ y \in X \end{matrix}$$

Then

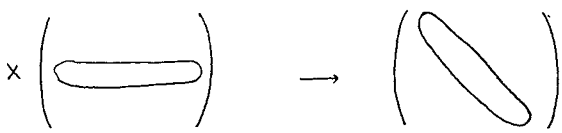
$$E_i^* E_j^* = \delta_{ij} E_i^*$$
$$E_0^* + \cdots + E_D^* = I$$
$$(E_i^*)^t = \overline{E_i^*}$$
$$= E_i^*$$

DEF. Dual Bose-Mesner algebra $M^* \equiv M^*(x)$
wrt $x$ is $\mathrm{Span}(E_0^*, \cdots, E_D^*)$.

Define dual associate matrices $A_0^*, \cdots, A_D^*$:
Indeed $A_i^* \equiv A_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$
is a diagonal matrix with

$$(A_i^*)_{yy} = |X| (E_i)_{xy} \qquad (y \in X)$$

$A_i^*$ is like a row $x$ of $E_i$




$E_i$ $\qquad\qquad$ $A_i^*$

Observe

$$A_i^* = \sum_{j=0}^{D} q_i(j) E_j^* \qquad \left( E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j \right)$$

$$E_i^* = \frac{1}{|X|} \sum_{j=0}^{D} p_i(j) A_j^* \qquad \left( A_i = \sum_{j=0}^{D} p_i(j) E_j \right)$$

So $A_0^*, \cdots, A_D^*$ form a basis for $M^*$

Also

$$A_i^* E_j^* = q_i(j) E_j^*$$

$$\left( A_i^* E_j^* = \sum_{j=0}^{D} q_i(j) E_j^* E_j^* = q_i(j) E_j^* \right)$$

So $q_i(j)$ are dual eigenvalues of $A_i^*$

Observe

$$A_0^* = I$$
$$A_0^* + \cdots + A_D^* = |X| E_0^*$$
$$\overline{A_i^*} = A_{\hat{i}}^*$$
$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \qquad (0 \le i, j \le D)$$

HS (pf $(A_0^*)_{yy} = |X| (E_0)_{xy} = (J)_{xy} = 1$

$$A_0^* + \cdots + A_D^* = \sum_{i=0}^{D} \sum_{j=0}^{D} q_i(j) E_j^* = |X| E_0^*$$

$(\because)$
$$I = E_0 + \cdots + E_D = \frac{1}{|X|} \sum_{i=0}^{D} \sum_{j=0}^{D} q_i(j) A_j$$

$$\sum_{i=0}^{D} q_i(j) = \delta_{j0} |X|$$

$$\overline{A_i^*} = \sum_{j=0}^{D} \overline{q_i(j)} E_j^* = \sum_{j=0}^{D} q_{\hat{i}}(j) E_j^* = A_{\hat{i}}^*$$

$$(A_i^* A_j^*)_{yy} = |X|^2 (E_i)_{xy} (E_j)_{xy}$$

$$= |X|^2 (E_i \circ E_j)_{xy} = |X| \sum_{h=0}^{D} q_{ij}^h (E_h)_{xy}$$

$$= \sum_{h=0}^{D} q_{ij}^h (A_h^*)_{yy} \quad )$$

LEMMA 35 Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme

Fix $x \in X$ $E_i^* \equiv E_i^*(x)$, $A_i^* \equiv A_i^*(x)$

(i) $E_i^* A_j E_k^* = 0$ iff $p_{ij}^k = 0$ $(0 \le i, j, k \le D)$

(ii) $E_i A_j^* E_k = 0$ iff $q_{ij}^k = 0$

Proofs will be given in Lecture 20 after a couple of lemmas

# Lecture 20 Mon. March 15 (Monday after spring break)

LEMMA 34-a Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme.

(i) $p_0(i) = 1$

(ii) $p_i(0) = k_i$, where

$$k_i = p_{ii'}^{0} = |\{y \in X \mid (x,y) \in R_i\}| \qquad (x \in X)$$

(iii) $q_0(i) = 1$

(iv) $q_i(0) = m_i$, where

$$m_i = \operatorname{rank} E_i$$

Proof.

(i)
$$\begin{aligned} A_0 &= p_0(0)E_0 + p_0(1)E_1 + \cdots + p_0(D)E_D \\ \| \\ I &= E_0 + E_1 + \cdots + E_D \end{aligned}$$

(ii)
$$A_i = p_i(0)E_0 + p_i(1)E_1 + \cdots + p_i(D)E_D$$

$$A_iE_0 = p_i(0)E_0 \qquad (E_0 = |X|^{-1}J)$$

$$A_iJ = p_i(0)J$$

$\uparrow$

has $k_i$ 1's in each row, so

$$A_iJ = k_iJ$$

$$\therefore\ k_i = p_i(0)$$

(iii)
$$\begin{aligned} E_0 &= |X|^{-1}(q_0(0)A_0 + q_0(1)A_1 + \cdots + q_0(D)A_D) \\ \| \\ |X|^{-1}J &= |X|^{-1}(A_0 + A_1 + \cdots + A_D) \end{aligned}$$

(iv)
$$E_i = |X|^{-1}(q_i(0)A_0 + q_i(1)A_1 + \cdots + q_i(D)A_D)$$

$E_i^2 = E_i$ and $E_i$ is similar to $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$

So

$$
\begin{aligned}
m_i &= \operatorname{rank} E_i \\
&= \operatorname{trace} E_i \\
&= \sum_{x \in X} (E_i)_{xx} \\
&= |X|\,|X|^{-1} q_i(0) \\
&= q_i(0)
\end{aligned}
$$

$$E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j$$

$$\downarrow$$

$$(E_i)_{xx} = \frac{1}{|X|} q_i(0) (A_0)_{xx}$$

LEMMA 34.b With the above notation.

(i) $p^h_{ij} = p^{h'}_{i'j'}$

(ii) $k_h p^h_{ij} = k_j p^j_{i'h} = k_i p^i_{hj'}$

(iii) $q^h_{ij} = q^{\hat{h}}_{\hat{i}\hat{j}}$

(iv) $m_h q^h_{ij} = m_j q^j_{\hat{i}h} = m_i q^i_{h\hat{j}}$

Proof

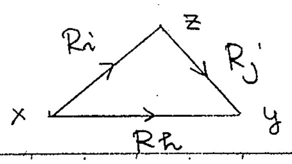
(i) $$(A_i A_j)^t = \left( \sum_{h=0}^{D} p^h_{ij} A_h \right)^t = \sum_{h=0}^{D} p^h_{ij} A_{h'}$$

$$= A_i^t A_j^t = A_{i'} A_{j'} = \sum_{h=0}^{D} p^{h'}_{i'j'} A_{h'}$$

(ii) $$\left| \{ xyz \in X^3 \mid (x,y) \in R_h \quad (x\,z) \in R_i,\ (z\,y) \in R_j \} \right|$$

$$= |X|\, k_h\, p^h_{ij} = |X|\, k_j\, p^j_{i'h} = |X|\, k_i\, p^i_{hj'}$$

x y z  z x y  x z y



(iii) $$(E_i \circ E_j)^t = \left( \frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^{h} E_h \right)^t = \frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^{h} E_{\hat{h}}$$

$$\| $$

$$E_{\hat{i}} \circ E_{\hat{j}}$$

$$= \frac{1}{|X|} \sum_{h=0}^{D} q_{\hat{i}\hat{j}}^{\hat{h}} E_{\hat{h}}$$

(iv) Let $\tau(B)$ denote the sum of the entries in the matrix $B$.

Observe : $\tau(B \circ C) = \operatorname{trace}(BC^t)$

Observe :

$$\tau(E_i \circ E_j \circ E_{\hat{k}}) = \tau((E_i \circ E_j \circ E_{\hat{k}})^t)$$

$$\begin{aligned}
\tau(E_i \circ E_j \circ E_{\hat{k}}) &= \operatorname{trace}((E_i \circ E_j) E_k) \\
&= \operatorname{trace}\left(\left(\frac{1}{|X|} \sum_h q_{ij}^{h} E_h\right) E_k\right) \\
&= \operatorname{trace}\left(\frac{1}{|X|} q_{ij}^{k} E_k\right) \\
&= |X|^{-1} m_k q_{ij}^{k}
\end{aligned}$$

$$\begin{aligned}
\tau(E_i \circ E_j \circ E_{\hat{k}}) &= \tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) \\
&= \operatorname{trace}((E_{\hat{i}} \circ E_k) E_j) \\
&= \operatorname{trace}\left(\frac{1}{|X|} q_{\hat{i}k}^{j} E_j\right) \\
&= |X|^{-1} m_j q_{\hat{i}k}^{j}
\end{aligned}$$

$$\begin{aligned}
\tau((E_i \circ E_j \circ E_{\hat{k}})^t) &= \tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}) \\
&= \operatorname{trace}((E_k \circ E_{\hat{j}}) E_i) \\
&= \operatorname{trace}\left(\frac{1}{|X|} q_{k\hat{j}}^{i} E_i\right) \\
&= |X|^{-1} m_i q_{k\hat{j}}^{i}
\end{aligned}$$

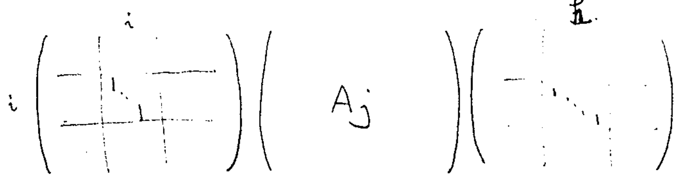
LEMMA 35 Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be any commutative scheme. Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$.

(i) $E_i^* A_j E_h^* = 0$ iff $p_{ij}^{h} = 0$

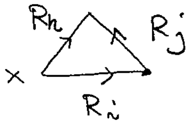
(ii) $E_i A_j^* E_h = 0$ iff $q_{ij}^{h} = 0$

$(0 \le i, j, h \le D)$

Proof (i).



$$= \begin{array}{c} \\ i \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} & h & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \boxed{*} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad (i\,h) \text{ block of } A_j$$

$= 0 \iff \nexists$  $\iff p^h_{ij} = 0$

(ii). The sum of the squares of norms of entries in $E_i A_j^* E_k$.

$$= \tau\left((E_i A_j^* E_k) \circ (\overline{E_i A_j^* E_k})\right)$$

$$= \mathrm{trace}\left(E_i A_j^* E_k (\overline{E_i A_j^* E_k})^t\right)$$

$$= \mathrm{trace}\left(E_i A_j^* E_k A_{\hat{j}}^* E_i\right) \qquad \mathrm{Trace}(XY) = \mathrm{Trace}(YX)$$

$$= \mathrm{trace}\left(E_i A_j^* E_k A_{\hat{j}}^*\right)$$

$$= \sum_{y \in X} \left(E_i A_j^* E_k A_{\hat{j}}^*\right)_{yy}$$

$$= \sum_{y \in X} \left( \sum_{z \in X} (E_i)_{yz} (A_j^*)_{zz} (E_k)_{zy} (A_{\hat{j}}^*)_{yy} \right)$$

$$(E_i)_{yz} \to (E_{\hat{i}})_{zy}, \quad (A_j^*)_{zz} \to |X|(E_j)_{xz}, \quad (A_{\hat{j}}^*)_{yy} \to |X|(E_{\hat{j}})_{xy} = |X|(E_j)_{yx}$$

$$= |X|^2 \left(E_j (E_{\hat{i}} \circ E_k) E_j\right)_{xx}$$

$$= |X|^2 \left(E_j \left(\frac{1}{|X|} \sum_{\ell=0}^{D} q^{\ell}_{\hat{i}k} E_\ell\right) E_j\right)_{xx}$$

$$= |X| \, q^j_{\hat{i}k} (E_j)_{xx} \qquad |X| E_j = \underline{q_j(0)} A_0 + q_j(1) A_1 + \cdots$$

$$= q^j_{\hat{i}k} m_j = m_k q^k_{ij} \qquad (E_j)_{xx} = \frac{1}{|X|} q_j(0) = \frac{m_j}{|X|}$$

COR 36. For any commutative scheme $Y=(X, \{R_i\}_{0\le i\le D})$ $q_{ij}^h$ is a non-negative real number $(0 \le h, i, j \le D)$

(Krein condition)

Proof.

$q_{ij}^k m_k$

is a nonnegative real by the proof of Lemma 35 (ii).
Also $m_k$ is a positive integer.

An interpretation of the Krein parameters.
Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a commutative scheme with standard module $V$.

Pick $v \in V$

$$v = \sum_{x\in X} \alpha_x \hat{x}$$

View $v$ as a function

$$X \to \mathbb{C}$$
$$(x \mapsto \alpha_x)$$

View $V$ as the set of all functions $X \to \mathbb{C}$.
Vector space $V$ together with product of functions is a $\mathbb{C}$-algebra

For $v = \sum_{x\in X} \alpha_x \hat{x}$ $\quad w = \sum_{x\in X} \beta_x \hat{x}$ $\quad \in V$

Write $$v \circ w := \sum_{x\in X} \alpha_x \beta_x \hat{x}$$

to represent the product of $v, w$ viewed as functions

LEMMA 37 With the above notation,

(i) $A_j^*(x)v = |X|(E_j\hat{x} \circ v)$ $\quad (\forall v \in V, \forall x \in X)$

(ii) $E_iV \circ E_jV \subseteq \sum_{\substack{h=0 \\ q_{ij}^h \neq 0}}^{D} E_hV$ $\quad (0 \leq i, j \leq D)$

(iii) $E_k(E_iV \circ E_jV) = E_kV$ $\quad$ if $q_{ij}^k \neq 0$

$(0 \leq k, i, j \leq D)$

## Lecture 21, Wed March 17, 1993

Proof of Lemma 37

(i) Suppose

$$v = \sum_{y \in X} \alpha_y \hat{y}. \qquad \text{Pick } z \in X.$$

Compare $z$ coordinate of each side in (i).

$$(A_j^*(x)v)_z = (A_j^*(x))_{zz} v_z$$

$$= |X|(E_j)_{xz}\,\alpha_z$$

$$|X|(E_{\hat{j}}\hat{x} \circ v)_z = |X| \underbrace{(E_{\hat{j}}\hat{x})_z}_{\substack{\text{column } x \text{ of } E_{\hat{j}} \\ = \text{row } x \text{ of } E_j}} \cdot \alpha_z$$

$$= |X|(E_j)_{xz}\,\alpha_z$$

(ii) Fix $i, j, k$ s.t. $q_{ij}^k = 0$.
show

$$0 \overset{?}{=} E_k(E_iV \circ E_jV)$$

$$= E_k(\text{Span}(v \circ w \mid v \in E_iV,\ w \in E_jV))$$

$$= E_k(\text{Span}(E_i\hat{y} \circ E_j\hat{z} \mid y, z \in X))$$

$$= \text{Span}(E_k(E_j\hat{z} \circ E_i\hat{y}) \mid y, z \in X)$$

$$= \text{Span}((E_k A_{\hat{j}}^*(z) E_i)\hat{y} \mid y, z \in X) \quad \text{by (i)}$$

But $q_{ij}^k = 0$ implies $q_{\hat{i}\hat{j}}^{\hat{k}} = 0$
So by Lemma 35 (ii)

$$0 = (E_{\hat{i}} A_{\hat{j}}^* E_{\hat{k}})^t = E_k A_{\hat{j}}^* E_i$$

Hence $E_k(E_iV \circ E_jV) = 0$

(iii) Fix $i, j, k$ s.t. $q_{ij}^{k} \neq 0$.

$E_k(E_iV \circ E_jV) \subseteq E_kV$ is clear

($\supseteq$) :

$$E_k(E_iV \circ E_jV) = E_k \operatorname{Span}(E_i\hat{y} \circ E_j\hat{z} \mid y, z \in X)$$

$$\supseteq E_k \operatorname{Span}(\underbrace{E_i\hat{y} \circ E_j\hat{y}} \mid y \in X)$$

(column $y$ of $E_i$) $\circ$ (column $y$ of $E_i$)

$= $ column $y$ of $E_i \circ E_i$

$= (E_i \circ E_i)\hat{y}$

$= \left( \frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^{h} E_h \right) \hat{y}$

$$= \operatorname{Span}(q_{ij}^{k} E_k \hat{y} \mid y \in X)$$

$$= \operatorname{Span}(E_k \hat{y} \mid y \in X)$$

since $q_{ij}^{k} \neq 0$

$$= E_kV$$

LEMMA 38 Given commutative scheme $Y=(X, \{R_i\}_{0\le i\le D})$

Fix $j$ $(0 \le j \le D)$

Define binary multiplication

$$\begin{array}{ccccc} E_jV & \times & E_jV & \longrightarrow & E_jV \\ v & & w & & v*w \end{array}$$

by $v*w = E_j(v \circ w)$.

Then

(i) $v*w = w*v$ $(\forall v, w \in E_jV)$

(ii) $v*(w+w') = v*w + v*w'$ $(\forall v, w, w' \in E_jV)$

(iii) $(\alpha v)*w = \alpha(v*w)$ $(\forall \alpha \in \mathbb{C})$

In particular, the vector space $E_jV$, together with $*$, is a commutative $\mathbb{C}$-algebra (not associative in general).

( $N_j$: $(E_jV, *)$ is called the Norton algebra on $E_jV$ )

(iv) $v*w=0$ $\forall v, w \in E_jV$ iff $q_{jj}^j = 0$.

Proof.

(i) – (iii) immediate.

(iv) immediate from Lemma 37 (ii) (iii).

Let $Y$, $j$, $N_j$ be as in Lemma 38

Let $\mathrm{Aut}\,Y = \{\sigma \in \mathrm{Mat}_X(\mathbb{C}) \mid \sigma$: permutation matrix that commutes with each element of Bose-Mesner algebra $M\}$

$= \{\sigma \in \mathrm{Mat}_X(\mathbb{C}) \mid \sigma$: permutation matrix $(x,y) \in R_i \to (\sigma x, \sigma y) \in R_i$ $\forall i,\ x, y \in X\}$

$\mathrm{Aut}(N_j) = \{\sigma : E_jV \to E_jV \mid \sigma$ is $\mathbb{C}$-algebra isom. i.e., $\sigma$ is an isomorphism of vector spaces and $\sigma(v * w) = \sigma(v) * \sigma(w)\ \forall v, w \in E_jV.\}$

LEMMA 39. Let $Y$, $j$, $*$ be as in Lemma 38.

(i) $E_jV$ is a module for $\mathrm{Aut}(Y)$.

(ii) $\sigma|_{E_jV} \in \mathrm{Aut}(N_j)$ $\quad \forall \sigma \in \mathrm{Aut}(Y)$

(iii)
$$\begin{array}{ccc} \mathrm{Aut}\,Y & \longrightarrow & \mathrm{Aut}(N_j) \\ \sigma & \longrightarrow & \sigma|E_j \end{array}$$
is a homomorphism of groups
(i.e., a representation of $\mathrm{Aut}(Y)$).

(iv) Suppose $R_0, \cdots, R_D$ are orbits of $\mathrm{Aut}(Y)$ acting on $X \times X$.

(so we are in Example 2 : Lecture 17 (17-5))

then above representation is irreducible.

Proof

(i) · Pick $\sigma \in \mathrm{Aut}\, Y$ $\quad v \in V$

$$\sigma E_j v = E_j \sigma v,$$

since $\sigma$ commutes with $M$.

(ii) $\sigma|E_jV : E_jV \to E_jV$

is an isomorphism of a vector space

since $\sigma$ is invertible.

$$\sigma(v * w) \overset{?}{=} \sigma(v) * \sigma(w) \qquad (v, w \in E_jV)$$

$$\| \qquad\qquad\qquad \|$$

$$\sigma(E_j(E_jv \circ E_jw))$$

$$\| \qquad\qquad (\sigma \text{ just permutes coordinates})$$

$$E_j\sigma(E_jv \circ E_jw) = E_j(E_j\sigma v \circ E_j\sigma w)$$

(iii) immediate from (i), (ii)

(iv) Here Bose-Mesner algebra $M$

is the full commuting algebra. i.e.,

$$M = \{ m \in \mathrm{Mat}_X(\mathbb{C}) \mid \sigma m = m\sigma \ \forall \sigma \in \mathrm{Aut}(Y) \}$$

Suppose

$\exists\, 0 \neq W \subsetneq E_jV$ that is $\mathrm{Aut}(Y)$-invariant.

Set

$$W^\perp := \{ v \in E_jV \mid \langle w, v \rangle = 0 \ \forall w \in W \}$$

Then $W^\perp$ is a module for $\mathrm{Aut}(Y)$

since $\mathrm{Aut}(Y)$ is closed under transpose conjugate.

Let

$e : V \to W$, $\quad f : V \to W^\perp$ orthogonal projection.

$$e + f = E_j$$

$e^2 = e$, $f^2 = f$, $ef = fe = 0$, $eE_k = 0$ if $k \neq j$.

$e$ commutes with all $\sigma \in Aut(Y)$
Hence $e \in M$.

$$e = \sum \alpha_i E_i$$

$\forall k \neq j$ :

$0 = e E_k$ So $\alpha_k = 0$

$$e = \alpha_j E_j$$

Hence $e = 0$ or $E_j$

i.e., $e = 0$ or $f = 0$. A contradiction.

Norton algebras were used in original construction of Monster finite simple group $G$.

Compute character table of $G$
→ $p_{ij}^h$, $q_{ij}^h$ of group scheme on $G$
→ find $j$ where $m_j = \dim E_j V$ is small
and $q_{jj}^j \neq 0$.
→ guess abstract structure of $N_j$
using knowledge of $p_{ij}^h$'s $q_{ij}^h$'s.
→ compute $Aut(N_j)$
→ $G$

Lecture 22, Fri. March 19, 1993

LEMMA 40 Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme

(i) $p^h_{0j} = p^h_{j0} = \delta_{jh}$

(ii) $p^0_{ij} = \delta_{ij'} k_i$

(iii) $q^h_{0j} = q^h_{j0} = \delta_{jh}$

(iv) $q^0_{ij} = \delta_{i\hat{j}} m_i$

(v) $\sum_{j=0}^{D} p^h_{ij} = k_i$

(vi) $\sum_{j=0}^{D} q^h_{ij} = m_i$

Proof

(i), (ii) trivial

(iii)
$$|X|^{-1} \sum_{l=0}^{D} q^l_{0j} E_l$$
$$= E_0 \circ E_j$$
$$= |X|^{-1} J \circ E_j$$
$$= |X|^{-1} \circ E_j$$

(iv) Recall
$$|X|^{-1} m_h q^h_{ij} = \tau(E_i \circ E_j \circ E_{\hat{h}}) \qquad \text{(Lemma 34b Lec 20-3)}$$
(where $\tau(B)$ is the sum of entries in matrix $B$)
$$|X|^{-1} m_0 q^0_{ij} = \tau(E_i \circ E_j \circ E_0) \qquad E_0 = \frac{1}{|X|} J$$
$$= |X|^{-1} \tau(E_i \circ E_j)$$
$$= |X|^{-1} \operatorname{trace}(E_i E_{\hat{j}})$$

$$= |X|^{-1} \delta_{i\hat{j}} \operatorname{trace} E_i$$

$$= |X|^{-1} \delta_{i\hat{j}} m_i$$

(v) Pick $x, y \in X$ $(x, y) \in R_h$

$$\sum_{j=0}^{D} p_{ij}^{h} = |\{z \in X \mid (x,z) \in R_i \quad (z,y) \in R_j \text{ for some } j\}|$$

$$= |\{z \in X \mid (x,z) \in R_i\}|$$

$$= k_i$$

(vi) $$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^{h} E_h$$

So

$$\sum_{j=0}^{D} E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} \left( \sum_{j=0}^{D} q_{ij}^{h} \right) E_h$$

$$= E_i \circ \sum_{j=0}^{D} E_j$$

$$= E_i \circ I$$

$$= |X|^{-1} \left( q_i(0) A_0 + q_i(1) A_1 + \cdots + q_i(D) A_D \right) \circ I$$

$$= |X|^{-1} q_i(0) I$$

$$= |X|^{-1} m_i (E_0 + E_1 + \cdots + E_D)$$

DEF. Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme

$Y$ is <u>Q-polynomial</u> w.r.t. ordering $E_0, \dots, E_D$ of primitive idempotents

if $q_{ij}^h \begin{cases} = 0 & \text{if one of } h.i.j \text{ is } > \text{ than sum of other 2} \\ \neq 0 & \text{if one of } h.i.j \text{ is } = \text{ the sum of other 2} \end{cases}$

In this case, set

$$c_i^* = q_{1\,i-1}^i, \quad a_i^* = q_{1\,i}^i \quad b_i^* = q_{1\,i+1}^i$$
$$(0 \le i \le D) \quad (c_0^* = b_D^* = 0)$$

Observe: Q-polynomial $\to$ $Y$: symmetric.

Suppose $i \neq \hat{i}$ for some $i$.

$$\begin{array}{l} q_{i\hat{i}}^0 = m_i \quad (\neq 0) \qquad \text{by Lemma 40 (iv)} \\ \parallel \\ 0 \qquad \text{by above inequality.} \end{array}$$

This is a contradiction

Hence

$$E_i^t = E_i \quad \text{for all } i.$$

Therefore

$M$ is symmetric and

$Y$ is symmetric

Observe: $Y$: Q-polynomial

$\to$ $c_i^* + a_i^* + b_i^* = m_1$ $(0 \le i \le D)$.

(just as $c_i + a_i + b_i = k$, for P-poly.)

By Lemma 40 (iv)

$$m_1 = \underset{\substack{\parallel \\ 0}}{q_{10}^i} + \underset{\substack{\parallel \\ 0}}{q_{11}^i} + \cdots + \underset{\substack{\parallel \\ c_i^*}}{q_{1\,i-1}^i} + \underset{\substack{\parallel \\ a_i^*}}{q_{1\,i}^i} + \underset{\substack{\parallel \\ b_i^*}}{q_{1\,i+1}^i} + \cdots$$

LEMMA 41. Assume $Y = (X, \{R_i\}_{0 \le i \le D})$ is symmetric scheme.

Pick $x \in X$, $E_i^* \equiv E_i^*(x)$, $A_i^* \equiv A_i^*(x)$

The following are equivalent

(i) $\Gamma$ is $Q$-polynomial w.r.t. $E_0, \cdots, E_D$.

(ii) $q_{1j}^h \begin{cases} = 0 & \text{if } |h-j| > 1 \\ \ne 0 & \text{if } |h-j| = 1 \end{cases} \quad (0 \le h, j \le D)$

(iii) $\exists f_i^* \in \mathbb{C}[\lambda]$, $\deg f_i^* = i$ and

$$A_i^* = f_i^*(A_1^*) \qquad (0 \le i \le D)$$

(iv) $E_0^*V, \cdots, E_D^*V$ are maximal eigenspaces of $A_1^*$, and

$$E_i A_1^* E_j = 0 \qquad \text{if } |i-j| > 1 \quad (0 \le i, j \le D)$$

(Compare (iv) with definition of $Q$-polynomial of Feb 1 Lecture 6 (Lec 6-8))

Proof

(i) → (ii) clear

(ii) → (iii)

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \qquad A_0^* = I \qquad \text{(Lec 19-7)}$$

$$A_1^* A_j^* = q_{1j}^{j-1} A_{j-1}^* + q_{1j}^j A_j^* + \underset{\ne 0}{q_{1j}^{j+1}} A_{j+1}^*$$

$$(1 \le j \le D-1)$$

Hence $A_j^*$ is a polynomial of degree exactly $j$ in $A_1^*$ by induction on $j$.

$$\lambda f_j^*(\lambda) = b_{j-1}^* f_{j-1}^*(\lambda) + a_j^* f_j^*(\lambda) + \underset{\ne 0}{c_{j+1}^*} f_{j+1}^*(\lambda)$$

$$f_{-1}^*(\lambda) = 0 \qquad f_0^*(\lambda) = 1$$

(iii) → (i)

Pick $i, j, h$ $(0 \le i, j, h \le D)$ $(h \ge i+j)$

Since

$m_h q^h_{ij} = m_j q^j_{ih} = m_i q^i_{hj}$ by Lemma 34 b,

it suffices to show that

$$q^h_{ij} \begin{cases} = 0 & \text{if } h > i+j \\ \neq 0 & \text{if } h = i+j. \end{cases}$$

$$A_i^* A_j^* = \sum_{h=0}^{D} q^h_{ij} A_h^*$$

$$f_i^*(A_1^*) f_j^*(A_1^*) = \sum_{h=0}^{D} q^h_{ij} f_h^*(A_1)$$

Hence

$$f_i^*(\lambda) f_j^*(\lambda) = \sum_{h=0}^{D} q^h_{ij} f_h^*(\lambda)$$

( $\because$ $A_0^*, A_1^*, \ldots, A_D^*$ are linearly indep.
So $f(A_1^*) = 0 \to \deg f > D$ )

$\deg \text{LHS} = i+j \to q^{i+j}_{ij} \neq 0$, $q^h_{ij} = 0$ if $h > i+j$.

(iii) → (iv)

Recall

$$A_1^* = q_1(0) E_0^* + q_1(1) E_1^* + \cdots$$

Each $A_i^*$ is a polynomial in $A_1^*$

Then $A_1^*$ generates the dual Bose-Mesner algebra

So

$q_1(0), q_1(1), \cdots q_1(D)$ are distinct

So $E_0^* V, \cdots, E_D^* V$

are maximal eigenspaces

Also $|i-j| > 1$, $q^j_{1i} = 0$

Thus $E_i A_1^* E_j = 0$ by Lemma 35 (ii)

(iv) → (ii)

$q_{1j}^{i} = 0$ if $|i-j|>1$, since in this case
$E_i A_1^* E_j = 0$ implies $q_{1j}^{i} = 0$
by Lemma 35 (ii)

Suppose $q_{1j}^{j+1} = 0$ for some $j$ $(0 \le j \le D-1)$.
WLOG choose $j$ min.
Then

$A_h^*$ is a polynomial of degree $h$
in $A_1^*$ $(0 \le h \le j)$

and

$$A_1^* A_j^* - q_{1j}^{j-1} A_{j-1}^* - q_{1j}^{j} A_j^* = 0.$$

LHS is a polynomial in $A_1^*$ of degree $j+1$.
Hence the minimal polynomial of $A_1^*$
has degree less than or equal to $j+1 \le D$
But $A_1^*$ has $D+1$ distinct eigenvalues.
This is a contradiction.

# Lecture 23 Mon. March 22, 1993

THEOREM 42 Let $Y=(X,\{R_i\}_{0\le i\le D})$ be a symmetric scheme. (View the standard module $V$ as an algebra of functions from $X\to\mathbb{C}$)
Then the following are equivalent
(i) $Y$ is Q-polynomial w.r.t. ordering $E_0,\dots,E_D$ of primitive idempotents.
(ii) $E_0V+E_1V+(E_1V)^2+\cdots+(E_1V)^i$
$=E_0V+E_1V+E_2V+\cdots+E_iV \qquad (0\le i\le D)$

Proof
By Lemma 37 (ii), (iii)
$$E_h(E_iV\circ E_jV)=0 \quad \text{iff} \quad q_{ij}^h=0$$
$$(0\le i,j,h\le D)$$

(i) → (ii)
By our assumption,
$q_{1j}^h=0$ if $|h-j|>1$. $q_{1j}^{j+1}\ne 0$.
So
$$E_1V\circ E_jV\subseteq E_{j-1}V+E_jV+E_{j+1}V \qquad (0\le j\le D) \text{ --- } (*)$$
$$E_{j+1}(E_1V\circ E_jV)=E_{j+1}V \qquad (0\le j\le D-1) \text{ --- } (**)$$
by Lemma 37.
Also $E_0V\in \mathrm{Span}(\delta)$ $\delta=$ all 1's vector
($=1$ as a function $X\to\mathbb{C}$)
So $E_0V\circ E_jV=E_jV \qquad (0\le j\le D)$ $(***)$
Show (ii) by induction on $i$.
$i=0,1$ Trivial
$i>1$ :
$\subseteq$: $E_0V+E_1V+(E_1V)^2+\cdots+(E_1V)^i$
$=E_0V+E_1V\circ(E_0V+E_1V+\cdots+(E_1V)^{i-1})$
$=E_0V+E_1V\circ(E_0V+E_1V+\cdots+E_{i-1}V)$
$\subseteq E_0V+E_1V+E_2V+\cdots+E_iV$ by $(*)$

$\supseteq$

Claim $\quad E_iV \subseteq E_1V \circ E_{i-1}V + E_{i-1}V + E_{i-2}V$
$(2 \le i \le D)$

Proof of Claim $\quad$ Since

$E_i(E_1V \circ E_{i-1}V) = E_iV \qquad (**)$

For $\forall v \in E_iV$,

$\exists u \in E_1V \circ E_{i-1}V$ s.t.

$E_iu = v$.

On the other hand

$E_1V \circ E_{i-1}V \subseteq E_{i-2}V + E_{i-1}V + E_iV \quad -(*)$

So $\quad u = w + v$, where $w \in E_{i-2}V + E_{i-1}V$

We have

$w = u - v \in E_1V \circ E_{i-1}V + E_{i-1}V + E_{i-2}V$.

as desired

$\boxed{\text{HS}}$ Note $\quad E_iV \circ E_jV = \text{Span}(u \circ v \mid u \in E_iV,\ v \in E_jV)$

$E_0V + E_1V + \cdots + E_iV \qquad$ By Claim

$\subseteq E_0V + E_1V + \cdots + E_{i-1}V + E_1V \circ E_{i-1}V$

$\subseteq E_0V + E_1V + \cdots + (E_1V)^{i-1} + E_1V(E_0V + E_1V + \cdots + (E_1V)^{i-1})$

$\subseteq E_0V + E_1V + \cdots + (E_1V)^{i-1} + (E_1V)^i$

(ii) $\to$ (i)

Claim 1 $\quad$ Pick $i, j \quad (0 \le i, j \le D) \quad j > i+1$

Then $q^j_{1i} = 0$.

Proof $\quad E_j(E_1V \circ E_iV) \subseteq E_j(E_1V \circ (E_0V + E_1V + \cdots + (E_1V)^i)$

$\subseteq E_j(E_0V + E_1V + \cdots + (E_1V)^{i+1})$

$= E_j(E_0V + E_1V + \cdots + E_{i+1}V)$

$= 0$

So $q^j_{1i} = 0$ by Lemma 37.

Claim 2 $\quad q_{1i}^{i+1} \neq 0 \quad (0 \leq i < D)$.

Proof $\quad E_0V + \cdots + E_{i+1}V$

$$= E_0V + E_1V + \cdots + (E_1V)^{i+1}$$

$$= E_0V + E_1V \circ (\underbrace{E_0V + \cdots + (E_1V)^{i}}_{E_0V + \cdots + E_iV})$$

$$= E_0V + E_1V \circ (E_0V + \cdots + E_iV)$$

So

$$E_{i+1}V = E_{i+1}(E_1V \circ (E_0V + \cdots + E_iV))$$

$$= E_{i+1}(E_1V \circ E_iV)$$

by Claim 1 and Lemma 37

Hence

$q_{1i}^{i+1} \neq 0$ by Lemma 37

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme with standard module $V$.

DEF. A representation of $Y$ is a pair $(\rho, H)$, where $H$ is a non-zero Hermitean space (with inner product $\langle\,,\,\rangle$) and $\rho : X \to H$ is a map s.t.

R1. $H = \mathrm{Span}(\rho(x) \mid x \in X)$

R2. $\langle \rho(x), \rho(y) \rangle$ depends only on $i$ for which $(x, y) \in R_i$ $\quad (x, y \in X)$

R3. $\forall x \in X \quad \forall i \quad (0 \leq i \leq D)$

$$\sum_{\substack{y \in X \\ (y,x) \in R_i}} \rho(y) \in \mathrm{Span}(\rho(x)).$$

Above reperesentation is nondegenerate if $\{\rho(x) \mid x \in X\}$ are distinct.

Example : $Y = H(D,2)$.

$X = \{ a_1 \cdots a_D \mid a_i \in \{1,-1\} \quad 1 \le i \le D \}$.

Let $H = \mathbb{C}^D$ $\langle \, . \, \rangle$ usual Hermitean dot product

For a vertex

$$x = a_1 \cdots a_D \in X,$$

define

$$\rho(x) = a_1 \cdots a_D \quad \text{vector in } H.$$

Then R1 — R3 hold.

$\boxed{HS}$ R1, R2 are obvious.

R3 W.m.a. $x = 1 \cdots 1$.

$$\sum_{y \in X \ (y,x) \in R_i} \rho(y)$$

restrict on the first coordinate

$-1$ appears $\binom{D-1}{i-1}$

$1$ appears $\binom{D-1}{i}$

So $$\sum_{y \in X \ (y,x) \in R_i} \rho(y) = \left( \binom{D-1}{i} - \binom{D-1}{i-1} \right) \rho(x).$$

Let $(\rho, H)$ be a representation of arbitrary Commutative scheme $Y$. Set

$$E := \left( \langle \rho(x), \rho(y) \rangle \right)_{x,y \in X}$$

Gram matrix of the representation

DEF. Representations $(\rho, H)$, $(\rho', H')$ of $Y$ are equivalent whenever Gram matrices are related by

$$E' \in \operatorname{Span} E$$

Do not distinguish between equivalent representations.

Note Suppose $(\rho, H)$ is a representation of a <u>symmetric</u> scheme $Y$.
Pick $x, y \in X$, $(x, y) \in R_j$.
Then $(y, x) \in R_j$. So

$$\langle \rho(x), \rho(y) \rangle = \langle \rho(y), \rho(x) \rangle \quad \text{by R2}$$
$$= \overline{\langle \rho(x), \rho(y) \rangle},$$

since $\langle\ ,\ \rangle$ is Hermitean
Hence the Gram matrix $E$ is real symmetric.
WLOG we can view $H$ as a real Euclidean space in this case.

LEMMA 43 Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme and $V$ a standard module.

Let $E_j$ be any primitive idempotent of $Y$.

(i) $(\rho, H)$ is a representation of $Y$, where $H = E_j V$ (with inner product inherited from $Y$).

$$\rho : X \longrightarrow H$$
$$x \longmapsto E_j \hat{x}$$

(i.e., $\rho(x)$ is the $x$-th column of $E_j$)

(ii) $\langle \rho(x), \rho(y) \rangle = |X|^{-1} q_j(i)$ if $(x, y) \in R_i$ $(x, y \in X)$

(iii) $$\sum_{y \in X,\ (y,x) \in R_i} \rho(y) = p_i(j) \rho(x).$$

$(0 \le i \le D,\ x, y \in X)$

(iv) $(\rho, H)$ is nondegenerate $\Leftrightarrow$ $q_j(i) \ne q_j(0)$ $(\forall i \in \{0, 1, \cdots, D\})$

(v). Every representation of $Y$ is equivalent to a representation of the above type for $j$ $(0 \le j \le D)$ and $j$ is unique.

Proof.

(i) – (iii)

R1 $\mathrm{Span}(\rho X)$ = the column space of $E_j$ $= H$.

R2 $$\begin{aligned}\langle \rho(x), \rho(y) \rangle &= \langle E_j \hat{x}, E_j \hat{y} \rangle \\ &= \overline{(E_j \hat{x})}^t E_j \hat{y} \\ &= \hat{x}^t \overline{E_j}^t E_j \hat{y} \\ &= \hat{x}^t E_j \hat{y} \\ &= (E_j)_{xy}\end{aligned}$$

$(\because {}^t\overline{E_j} = E_j$ Lemma 34 Lec 19-1 $)$

Recall
$$E_j = |X|^{-1}( q_j(0) A_0 + \cdots + q_j(D) A_D )$$
So
$$(E_j)_{xy} = |X|^{-1} q_j(i) \text{ , where } (x,y) \in R_i .$$

R3. Recall
$$A_i = p_i(0) E_0 + \cdots + p_i(D) E_D .$$
So
$$E_j A_i = p_i(j) E_j ,$$
$$E_j A_i \hat{x} = p_i(j) E_j \hat{x} = p_i(j) \rho(x)$$
$$\| $$
$$E_j \sum_{y \in X \ (y,x) \in R_i} \hat{y}$$
$$= \sum_{y \in X, \ (y,x) \in R_i} \rho(y)$$

Note. $$A_i \hat{x} = \sum_{y \in X \ (x,y) \in R_{i'}} \hat{y}$$ ← $\boxed{HS}$ Trivial

(pf.) z entry of LHS
$$= (A_i \hat{x})_z$$
$$= \sum_{w \in X} (A_i)_{zw} \hat{x}_w$$
$$= (A_i)_{zx}$$
$$= \begin{cases} 1 & \text{if } (x,z) \in R_{i'} \\ 0 & \text{else} \end{cases}$$
z entry of RHS
$$= \sum_{y \in X \ (x,y) \in R_{i'}, \ z = y} 1$$
$$= \begin{cases} 1 & \text{if } (x,y) \in R_{i'} \\ 0 & \text{else} \end{cases}$$

(iv). By (ii).

$$\|\rho(x)\|^2 = \langle \rho(x), \rho(x) \rangle$$
$$= |X|^{-1} q_j(0)$$
$$= |X|^{-1} m_j \qquad (m_j = \dim E_j V)$$

independent of $x \in X$.

Pick distinct $x, y \in X$ s.t.

$(x, y) \in R_i \qquad (i \neq 0)$

Then

$$\rho(x) = \rho(y)$$
$$\Leftrightarrow \langle \rho(x), \rho(y) \rangle = \|\rho(x)\|^2 = |X|^{-1} q_j(0)$$
$$\Leftrightarrow |X|^{-1} q_j(i) = |X|^{-1} q_j(0)$$
$$\Leftrightarrow q_j(i) = q_j(0).$$

Lecture 24 Wed. March 23, 1993

No Class on Friday (another conference)

(Proof of LEMMA 43 continued)

$E_j$ : primitive idempotent

$H = E_j V$ $\qquad \rho : X \to H \quad (x \mapsto E_j \hat{x})$

(v) Every representation $(\rho, H)$ of $Y$ is equivalent to a representation of above type, for some $j$ $(0 \le j \le D)$ (and $j$ is unique):

Let $\quad E := (\langle \rho(x), \rho(y) \rangle)_{x, y \in X}$.

By R2

$$E = \sum_{i=0}^{D} \sigma_i A_i \qquad \text{some } \sigma_0, \cdots, \sigma_D \in \mathbb{C}.$$

Hence $E$ belongs to the Bose-Mesner algebra $M$ of $Y$.

We want to show that $E$ is a scalar multiple of a primitive idempotent.

Fix $x \in X$ and fix $i$ $(0 \le i \le D)$

By R3

$$\sum_{\substack{y \in X \\ (y,x) \in R_i}} \rho(y) = \alpha \rho(x) \qquad \text{some } \alpha \in \mathbb{C}. \qquad -(*)$$

So

$$\underbrace{\langle \sum_{y \in X, (y,x) \in R_i} \rho(y), \rho(x) \rangle}_{= k_i \overline{\sigma_i}} = \bar{\alpha} \underbrace{\langle \rho(x), \rho(x) \rangle}_{= \alpha \sigma_0}$$

Hence $\alpha$ is independent of $x$.

In matrix form $(*)$ becomes

$$E A_i \hat{x} = \alpha E \hat{x}$$

$\boxed{\text{HS}}$ $\quad Eu = Ev \iff \langle z, Eu \rangle = \langle z, Ev \rangle \quad \forall z \in X$

$$\iff (Eu)_z = (Ev)_z \quad \forall z \in X$$

$$(EA_i\hat{x})_z = \langle \rho(z), \sum_{y \in X,\ (y,x) \in R_i} \rho(y) \rangle$$

$$= \alpha \langle \rho(z), \rho(x) \rangle$$

$$= (\alpha E\hat{x})_z$$

Hence $\quad EA_i\hat{x} = \alpha E\hat{x}$

Since $x$ is arbitrary,

$$EA_i = \alpha E$$

So $\quad EA_i \in \operatorname{Span} E \quad$ and

$$EM = \operatorname{Span} E$$

We have $\quad E \in \operatorname{Span}(E_j)$ for unique $j$ $(0 \le j \le D)$

$\boxed{\text{HS}}$ $\quad E = \tau_0 E_0 + \cdots + \tau_D E_D \qquad \tau_j \in \mathbb{C}\ (0 \le j \le D)$

at least one of $\tau_j$ is nonzero

$$\tau_j E_j = EE_j \in \operatorname{Span} E$$

So $\quad \tau_j E_j = E$

as $E_0, \ldots, E_D$ are linearly independent.

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a symmetric scheme and let $E$ be a primitive idempotent.

DEF $Y$ is Q-polynomial w.r.t. $E$
$\Leftrightarrow$ $Y$ is Q-polynomial w.r.t some ordering $E_0, E_1, \cdots, E_D$ of primitive idempotents, where $E_0 = |X|^{-1}J$. and $E_1 = E$.

THEOREM 44. Assume $Y = (X, \{R_i\}_{0 \le i \le D})$ is P-polynomial (i.e., $(X, R_1)$ is distance-regular)
Let $E$ be any primitive idempotent of $Y$.
Let $(\rho, H)$ be the corresponding representation.

(i) The following are equivalent.

(ia) $Y$ is Q-polynomial w.r.t. $E$.

(ib) $(\rho, H)$ is nondegenerate and $\forall x, y \in X$ $\forall i, j$ $(0 \le i, j \le D)$, HS $\boxed{\theta_1^* \neq \theta_2^*}$

$$\sum_{\substack{z \in X,\ (x,z) \in R_i \\ (y,z) \in R_j}} \rho(z) - \sum_{\substack{z' \in X\ (x,z) \in R_j \\ (y,z) \in R_i}} \rho(z') \in \mathrm{Span}(\rho(x) - \rho(y))$$

(ic) $(\rho, H)$ is nondegenerate and $\forall x, y \in X$ HS $\boxed{\theta_1^* \neq \theta_2^*}$

$$\sum_{\substack{z \in X\ (x,z) \in R_1 \\ (y,z) \in R_2}} \rho(z) - \sum_{\substack{z' \in X,\ (x,z) \in R_2 \\ (y,z) \in R_1}} \rho(z') \in \mathrm{Span}(\rho(x) - \rho(y))$$

(ii) Write

$$E = |X|^{-1} \sum_{j=0}^{D} \theta_j^* A_j .$$

and suppose (ia)-(ic) hold, then the coefficient in (ib) is

$$p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \qquad (1 \le h \le D,\ \ 0 \le i, j \le D)$$

Proof

(ia) → (ib)  WLOG $E \equiv E_1$

$Y$ is $Q$-polynomial w.r.t $E$

($\boxed{\text{HS}}$ Then by Lemma 41 (Lec22-4))

$\theta_0^*, \ldots, \theta_D^*$ are distinct.

So $\theta_h^* \neq \theta_0^*$ $\forall h \in \{1, 2, \ldots, D\}$

and $(p, H)$ is nondegenerate.

Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$,

$A_i^* \equiv A_i^*(x)$, $A^* \equiv A_1^*$.

Let $M$ be the Bose-Mesner algebra.

Set $L = \{ m A^* n - n A^* m \mid n, m \in M \}$

Claim 1  $\dim L \leq D$.

Proof of Claim 1

$$L = \mathrm{Span}(E_i A^* E_j - E_j A^* E_i \mid 0 \leq i < j \leq D)$$

$$= \mathrm{Span}(E_i A^* E_{i+1} - E_{i+1} A^* E_i \mid 0 \leq i \leq D-1),$$

Since $E_i A^* E_j = 0$ if $q_{ij}^1 = 0$,

(Lemma 35 Lec19-8 Lemma 34 b Lec 20-2)

and this occurs if

$|i-j| > 1$ by $Q$-polynomial property.

Hence $\dim L \leq D$.

Claim 2 (i) $\{ A^* A_h - A_h A^* \mid 1 \leq h \leq D \}$ is a basis for $L$. In particular,

(ii) $\exists\, r_{ij}^h \in \mathbb{C}$ $(1 \leq h \leq D,\ 0 \leq i, j \leq D)$, s.t.

$$A_i A^* A_j - A_j A^* A_i = \sum_{h=1}^{D} r_{ij}^h (A^* A_h - A_h A^*)$$

Proof of Claim 2

(i) The column $x$ of $A^*A_h - A_hA^*$

$$\left[\ \boxed{HS}\quad ((A^*A_h - A_hA^*)\hat{x})_y = E_{xy}(A_h)_{yx} - (A_h)_{yx}E_{xx} = (\theta_h^* - \theta_0^*)(A_h)_{yx}\ \right]$$

is a non zero scalar

$\theta_h^* - \theta_0^*$

times the column $x$ of $A_h$

Also the column $x$ of $A_0, A_1, \ldots, A_D$ are linearly independent.

Hence the matrices given are linearly independent.

They are in $L$ by construction, so they form a basis for $L$ by Claim 1.

(ii) This is immediate since

$$A_iA^*A_j - A_jA^*A_i \in L \qquad \forall i,j.$$

Claim 3
$$r_{ij}^\ell = p_{ij}^\ell\left(\frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_\ell^*}\right) \qquad (1 \le \ell \le D,\ 0 \le i,j \le D)$$

Proof of Claim 3 Fix $i, j$

$$A_iA^*A_j - A_jA^*A_i - \sum_{h=1}^{D} r_{ij}^h(A^*A_h - A_hA^*) = 0.$$

Pick $\ell$ $(1 \le \ell \le D)$, Pick $y \in X$, s.t. $(x,y) \in R_\ell$.

$$(A_iA^*A_j)_{xy} = \sum_{z\in X}(A_i)_{xz}(A^*)_{zz}(A_j)_{zy}$$

$$= \sum_{\substack{z\in X\\ (x,z)\in R_i\ (y,z)\in R_j}} (A^*)_{zz}$$

$$= |X|^{-1}p_{ij}^\ell\,\theta_i^*.$$

Similarly $(A_jA^*A_i)_{xy} = |X|^{-1}p_{ij}^\ell\,\theta_j^*$

$$(A^*A_h - A_hA^*)_{xy} = (A_0A^*A_h - A_hA^*A_0)_{xy}$$
$$= |X|^{-1} p^l_{0h} (\theta^*_0 - \theta^*_h)$$
$$= \begin{cases} 0 & \text{if } l \neq h \\ |X|^{-1}(\theta^*_0 - \theta^*_h) & \text{if } l = h \end{cases}$$

Hence $\sum_{h=1}^{D} r^h_{ij} (A^*A_h - A_hA^*)_{xy} = |X|^{-1} r^l_{ij} (\theta^*_0 - \theta^*_l)$

Comparing terms, we have

$$p^l_{ij}(\theta^*_i - \theta^*_j) - r^l_{ij}(\theta^*_0 - \theta^*_l) = 0$$

Claim 4 $\forall h\ (1 \le h \le D)$, $\forall i,j\ (0 \le i,j \le D)$, $\forall w, y \in X$, $(w,y) \in R_h$

$$\sum_{(w,z)\in R_i\ (y,z)\in R_j} p(z) - \sum_{(w,z)\in R_j,\ (y,z)\in R_i} p(z') - r^h_{ij}(p(w) - p(y)) = 0 \quad (\text{✡})$$

Proof of Claim 4

It suffices to show that

$\langle$ LHS of (✡), $p(x) \rangle = 0$

(Since $x$ is arbitrary, if LHS of (✡) is orthogonal to all vertices in H, LHS of ✡ $= 0$ )

$\langle$ LHS of ✡, $p(x) \rangle$

$$= \sum_{\substack{z\in X \\ (w,z)\in R_i \\ (y,z)\in R_j}} \underbrace{\langle p(z), p(x)\rangle}_{E_{zx} = |X|^{-1}A^*_{zz}} - \sum_{\substack{z'\in X \\ (w,z')\in R_j \\ (y,z')\in R_i}} \langle p(z'), p(x)\rangle - r^h_{ij}\langle p(w)-p(y), p(x)\rangle$$

$$|X|^{-1}\sum_{z\in X} (A_i)_{wz}(A^*)_{zz}(A_j)_{zy}$$

$$= |X|^{-1}(A_iA^*A_j)_{wy} - |X|^{-1}(A_jA^*A_i)_{wy} - |X|^{-1}\sum_{l=1}^{D} r^l_{ij}(A^*A_l - A_lA^*)_{wy}$$

$= |X|$ times $w,y$ entry of a matrix known to be 0 by Claim 2.

$= 0$.

$$|X|^{-1}A^*_{ww} = E_{xw}$$

$$\boxed{\text{HS}}\ |X|^{-1}\sum_{l=1}^{D} r^l_{ij}(A^*A_l - A_lA^*)_{wy} = |X|^{-1} r^h_{ij} (A^*A_h - A_hA^*)_{wy} = r^h_{ij}(\langle p(x), p(w)\rangle - \langle p(x), p(y)\rangle)$$

# Lecture 25 Mon. March 29

(ib) → (ic) Obvious.

(ic) → (ia) WLOG $D \geq 3$. else trivial

( $\boxed{HS}$ The case $D=2$ should be treated somewhere, but the assumption $D \geq 3$ is not used. )

Fix $w \in X$, and write $E_i^* \equiv E_i^*(w)$ $A_i^* \equiv A_i^*(w)$, $A^* \equiv A_1^*$

$A_i$ : $i$-th distance matrix.

$$E \equiv E_1 = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i .$$

Since $(\rho, H)$ is nondegenerate,

$\theta_0^* \neq \theta_h^*$ $\forall h \in \{1, 2, \cdots, D\}$ (Lemma 43 (iv))

<u>Claim 1</u> Pick $h$ $(1 \leq h \leq D)$, and $x, y$ with $(x,y) \in R_h$.

Then

$$\sum_{\substack{z \in X \\ (x z) \in R_1,\ (y z) \in R_2}} \rho(z) - \sum_{\substack{z' \in X \\ (x z') \in R_2,\ (y, z') \in R_1}} \rho(z') = r_{12}^h (\rho(x) - \rho(y)),$$

where $$r_{12}^h = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}$$

<u>Proof of Claim 1</u>

By our assumption

$$\sum_{\substack{z \in X \\ (x z) \in R_1,\ (y z) \in R_2}} \rho(z) - \sum_{\substack{z' \in X \\ (x z') \in R_2\ (y, z') \in R_1}} \rho(z') = \alpha\, (\rho(x) - \rho(y))$$

Hence

$$\Big\langle \sum_{\substack{z \in X \\ (x z) \in R_1,\ (y z) \in R_2}} \rho(z) - \sum_{\substack{z' \in X \\ (x z') \in R_2,\ (y, z') \in R_1}} \rho(z'),\ \rho(x) \Big\rangle = \alpha \langle \rho(x) - \rho(y),\ \rho(x) \rangle$$

$$\| \qquad (\bar{\alpha} = \alpha \text{ as } E \text{ is symmetric}) \qquad \|$$

$$|X|^{-1} p_{12}^h (\theta_1^* - \theta_2^*) \qquad\qquad \alpha |X|^{-1} (\theta_0^* - \theta_h^*)$$

We have

$$\alpha = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}$$

Claim 2 $\quad A_1A^*A_2 - A_2A^*A_1 = \sum_{h=1}^{D} r_{12}^h (A^*A_h - A_hA^*)$

Proof of Claim 2 $\quad$ The $xy$ entry of the LHS − RHS is

$$|X| \left\langle \sum_{\substack{z\in X \\ (xz)\in R_1 \\ (yz)\in R_2}} \rho(z) - \sum_{\substack{z'\in X \\ (xz')\in R_1 \\ (y,z')\in R_2}} \rho(z') - r_{12}^h(\rho(x)-\rho(y)),\ \rho(w) \right\rangle$$

where $(xy) \in R_h \quad h = 1, 2, \ldots, D$

and the $xy$ entry of the LHS − RHS is 0 if $x = y$.

But the vector on the left in the above inner product is 0 by Claim 1 so the inner product is 0.

Thus the $xy$ entry of the LHS − RHS is always 0. and we have Claim 2

Claim 3 $\quad A^*A_3 - A_3A^* \in \mathrm{Span}(AA^*A_2 - A_2A^*A,\ A^*A_2 - A_2A^*,\ A^*A - AA^*)$

Proof of Claim 3 $\qquad p_{12}^h = 0$ if $h > 3$

$\qquad\qquad\qquad\qquad \neq 0$ if $h = 3$

We have $\qquad r_{12}^h = 0$ if $h > 3$

$\qquad\qquad\qquad\qquad \neq 0$ if $h = 3$ $\qquad \boxed{\theta_1^* \neq \theta_2^*}$

Now done by Claim 2

Claim 4 $\quad \exists\, \beta, \gamma, \delta \in \mathbb{R}$ s.t

(i) $\quad 0 = [A,\ A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*]$

$$= A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)$$

Proof of Claim 4

There exists $f_i \in \mathbb{R}[\lambda]$ $\deg f_i = i$ s.t. $A_i = f_i(A_1)$.

Writing $A_2, A_3$ as polynomials in $A$ in Clai 3 and simplifying. we find

$A^3A^* - A^*A^3 \in \mathrm{Span}(A^2A^*A - AA^*A^2,\ A^2A^* - A^*A^2,\ AA^* - A^*A)$

$\boxed{\text{HS}}$ $A_3 = \beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I \qquad \beta_3 \neq 0$

$A_2 = \gamma_2 A^2 + \gamma_1 A + \gamma_0 I \qquad \gamma_2 \neq 0$

$$A^*A_3 - A_3A^*$$
$$= A^*(\beta_3A^3 + \beta_2A^2 + \beta_1A + \beta_0 I) - (\beta_3A^3 + \beta_2A^2 + \beta_1A + \beta_0 I)A^*$$
$$A^3A^* - A^*A^3$$
$$\in \mathrm{Span}(A^*A_3 - A_3A^*,\ A^2A^* - A^*A^2,\ AA^* - A^*A)$$
$$\subseteq \mathrm{Span}(AA^*A_2 - A_2A^*A,\ A^*A_2 - A_2A^*,\ A^2A^* - A^*A^2,\ AA^* - A^*A)$$
$$A^*A_2 - A_2A^*$$
$$= A^*(\gamma_2A^2 + \gamma_1A + \gamma_0 I) - (\gamma_2A^2 + \gamma_1A_1 + \gamma_0 I)A^*$$
$$AA^*A_2 - A_2A^*A$$
$$= AA^*(\gamma_2A^2 + \gamma_1A + \gamma_0 I) - (\gamma_2A^2 + \gamma_1A_1 + \gamma_0 I)A^*A$$
$$\therefore\ A^*A_2 - A_2A^*$$
$$\in \mathrm{Span}(A^2A^* - A^*A^2,\ AA^* - AA^*)$$
$$AA^*A_2 - A_2A^*A$$
$$\in \mathrm{Span}(A^2A^*A - AA^*A^2,\ AA^* - A^*A)$$
$$\therefore\ A^3A^* - A^*A^3$$
$$\in \mathrm{Span}(A^2A^*A - AA^*A^2,\ A^2A^* - A^*A^2,\ AA^* - A^*A)$$

Hence we can find $\beta, \gamma, \delta$ satisfying

$$0 = A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)$$

On the other hand

$$[A,\ A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*]$$

$$= A^3A^* - A^2A^*A - \beta A^2A^*A + \beta AA^*A^2 + AA^*A^2 - A^*A^3$$
$$- \gamma A^2A^* - \cancel{\gamma AA^*A} + \cancel{\gamma AA^*A} + \gamma A^*A^2 - \delta AA^* + \delta A^*A$$
$$= A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)$$

Thus we have (i) and (ii).

Define a diagram $D_E$ on nodes $0, 1, \dots, D$.
Connect distinct nodes $i, j$ by undirected arc if $q^1_{ij} \neq 0$. (Note $q^1_{ij} = q^1_{ji}$).
Since $q^1_{0j} = \delta_{1j}$, the 0-node is adjacent to the 1-node and no other node.

$Y$ is $Q$-polynomial wrt $E$ $\leftrightarrow$ $D_E$ is a path.

Claim 5 $D_E$ is connected

Proof of Claim 5

Suppose $\exists \Delta \subseteq \{0, 1, \dots, D\}$ s.t.
$i, j$ not connected $\forall i \in \Delta$ $\forall j \in \{0, 1, \dots, D\} - \Delta$.
Set $$f = \sum_{i \in \Delta} E_i.$$

Observe: $$fA^* = \sum_{i \in \Delta} E_i A^* \left( \sum_{j=0}^{D} E_j \right)$$
$$= \sum_{i \in \Delta, j \in \Delta} E_i A^* E_j \quad \left( \begin{array}{c} \text{Since } E_i A^* E_j = 0 \\ \text{if } q^1_{ij} = 0 \end{array} \right)$$
$$= f A^* f$$

Also $A^* f = f A^* f$.
Hence $f$ commutes with $A^*$.
But $f$ is an element of the Bose-Mesner algebra
$$f = \sum_{i=0}^{D} \alpha_i A_i \quad \text{for some } \alpha_0, \dots, \alpha_D \in \mathbb{C}$$

We have
$$0 = fA^* - A^* f = \sum_{i=1}^{D} \alpha_i (A_i A^* - A^* A_i)$$

But $\{ A_h A^* - A^* A_h \mid 1 \le h \le D \}$ are linearly independent
[ the column $\omega$ of $A_h A^* - A^* A_h$ is $\theta^*_h - \theta^*_0$ times the column $\omega$ of $A_h$ ]

HS See: L24-5

Hence $\alpha_1 = \cdots = \alpha_D = 0$

and $f = \alpha_0 I$.

Since $f^2 = f$, $\alpha_0 = 0$ or $1$.

If $\alpha_0 = 0$, $f = 0$ and $\Delta = \phi$

If $\alpha_0 = 1$. $f = I$ and $\Delta = \{0, 1, \cdots, D\}$

This proves Claim 5.

$\boxed{\text{HS}}$ Claim 5 proves the following in general

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a symmetric association scheme. Fix $x \in X$ and let

$$E = \frac{1}{|X|} \sum_{j=0}^{D} \theta_j^* A_j \qquad (\theta_j^* = q_1(j) \text{ if } E = E_1)$$

be a primitive idempotent and $E_j^* \equiv E_j^*(x)$.

$$A^* = \sum_{j=0}^{D} \theta_j^* E_j^*$$

If $\theta_0^* \neq \theta_h^*$ $h = 1, \cdots, D$, then the following hold

(i) $\{ A_h A^* - A^* A_h \mid 1 \le h \le D \}$ are linearly independent

(ii) The diagram $D_E$ on nodes $0, 1, \cdots, D$ defined by

$$i \sim j \Leftrightarrow E(E_i \circ E_j) \neq 0$$

is connected.

(iii) $C_M(A^*) = \{ L \in M \mid LA^* = A^* L \} = \mathrm{Span}(I)$

Proof (i) The column $x$ of $A_h A^* - A^* A_h$ is $(\theta_0^* - \theta_h^*)$ times the column $x$ of $A_h$.

(iii) $0 = [\sum_{h=0}^{D} \alpha_h A_h, A^*] = \sum_{h=1}^{D} \alpha_h (A_h A^* - A^* A_h)$ $\therefore$ $\alpha_1 = \cdots = \alpha_D = 0$

(ii) $\Delta$: connected component $f = \sum_{i \in \Delta} E_i$, then $f \in C_M(A^*)$

◎ $Y = (X, \{R_i\}_{0 \le i \le 2})$ : symmetric association scheme with $D=2$ Let

$$E = \frac{1}{|X|} \sum_{j=0}^{2} \theta_j^* A_j$$

be a primitive idempotent. If $\theta_0^* \neq \theta_1^*, \theta_2^*$ Then $Y$ is $Q$-polynomial w.r.t. $E$.

Proof. By the previous lemma, $D_E$ is connected.

○ It seems $\theta_1^* \neq \theta_2^*$ is necessary.
Clarify the condition $\theta_1^* = \theta_2^*$

・ Terwilliger claims that $\theta_1^* = \theta_2^*$ does not occur under the assumption of (i'c) (March 7, 1995)

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Assume $Y = (X, \{R_i\}_{0 \le i \le D})$ is P-polynomial.
Let $E$ be a primitive idempotent of $Y$.
s.t. corresponding representation $(\rho, H)$ is nondegenerate.
Show for $\forall x, y \in X$

$$\sum_{\substack{z \in X,\ (x,z) \in R_1 \\ (y,z) \in R_2}} \rho(z) - \sum_{\substack{z' \in X\ (x,z') \in R_2 \\ (y,z') \in R_1}} \rho(z') \in \mathrm{Span}(\rho(x) - \rho(y))$$

implies that $Y$ is Q-polynomial w.r.t. $E$.

Define a diagram $D_E$ on nodes $0, 1, \cdots, D$
$i - j \iff q'_{ij} \neq 0 \quad i \neq j$
by setting $E = E_1$.

We showed that $0 - j \leftrightarrow j = 1 \quad (1 \le j \le D)$
and $D_E$ is connected.

Now it is sufficient to show the following.
Claim 6 Let $i$ be a node in $D_E$.
Then $i$ is adjacent to at most 2 arcs.
Proof of Claim 6
Suppose the node $j$ is adjacent to $i$ in $D_E$.
By Claim 4

$$0 = E_i\left(A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)\right)E_j$$

$$= E_iA^*E_j\left(\theta_i^3 - \theta_j^3 - (\beta+1)(\theta_i^2\theta_j - \theta_i\theta_j^2) - \gamma(\theta_i^2 - \theta_j^2) - \delta(\theta_i - \theta_j)\right)$$

$$= E_iA^*E_j\,(\theta_i - \theta_j)\,p(\theta_i, \theta_j)$$

where $p(s,t) = s^2 - \beta st + t^2 - \gamma(s+t) - \delta$.

$\boxed{\text{HS}}$ $(\theta_i - \theta_j)(\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \delta)$
$= (\theta_i^3 - \theta_j^3 - (\beta+1)(\theta_i^2\theta_j - \theta_i\theta_j^2) - \gamma(\theta_i^2 - \theta_j^2) - \delta(\theta_i - \theta_j))$

Since $i$ is adjacent to $j$, $g_{ij} \neq 0$ and

$E_i A^* E_j \neq 0$ by Lemma 35 (ii) (Lec 19-8)

Since $Y$ is P-polynomial,

$\theta_i \neq \theta_j$ if $i \neq j$.

Hence $p(\theta_i, \theta_j) = 0$.

But $p$ is quadratic in $t$.

So $p(\theta_i, t) = 0$

has at most 2 solutions for $\theta_j$.

Now $D_E$ is a path, and $\Gamma$ is $Q$-polynomial w.r.t $E$. This proves Thm 4.4.

Corollary 45 Assume $Y = (X, \{R_i\}_{0 \le i \le D})$ is P-polynomial, and Q-polynomial w.r.t. a primitive idempotent

$$E = \frac{1}{|X|} \sum_{i=0}^{D} \theta_i^* A_i.$$

Then
$$\beta = \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*}$$
is independent of $i$. $(0 \le i \le D-3)$

Proof. Fix $i$. WLOG $D \ge 3$ else vacuous.

Pick $x, y \in X$ with $(x, y) \in R_3$.

Let $(\rho, H)$ be the representation for $E$.

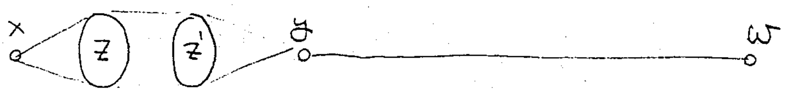
Then
$$\sum_{\substack{z \in X \\ (x,z) \in R_1, (y,z) \in R_2}} \rho(z) - \sum_{\substack{z' \in X \\ (x,z') \in R_2, (y,z') \in R_1}} \rho(z') = \frac{p^3_{12}(\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_3^*}(\rho(x) - \rho(y)) \quad -(*)$$

and $p^3_{1,2} = c_3$.

Since $p^3_{i,i+3} \neq 0$, there exists $w \in X$

s.t. $(x, w) \in R_{i+3}$ $(y, w) \in R_i$.

Take inner product of $*$ with $\rho(w)$




$P^3_{12}(x,y)$ $P^3_{21}(x,y)$

$$P^3_{12}(x,y) \subset P^{i+3}_{1\ i+2}(x,w) \cap P^{i}_{2\ i+2}(y,w)$$
$$P^3_{21}(x,y) \subset P^{i+3}_{2\ i+1}(x,w) \cap P^{i}_{1\ i+1}(y\,w)$$

Hence

$$\left\langle \sum_{\substack{z\in X\\ (x,z)\in R_1\ (y,z)\in R_2}} \rho(z) - \sum_{\substack{z'\in X\\ (x,z')\in R_2\ (y,z)\in R_1}} \rho(z'),\ \rho(w) \right\rangle$$
$$= |X|^{-1} c_3 (\theta^*_{i+2} - \theta^*_{i+1})$$

$$\left\langle \frac{c_3(\theta^*_1 - \theta^*_2)}{\theta^*_0 - \theta^*_3} (\rho(x) - \rho(y)),\ \rho(w) \right\rangle$$
$$= \frac{c_3(\theta^*_1 - \theta^*_2)}{\theta^*_0 - \theta^*_3} |X|^{-1} (\theta^*_{i+3} - \theta^*_i)$$

We have

$$\sigma = \frac{\theta^*_{i+1} - \theta^*_{i+2}}{\theta^*_i - \theta^*_{i+3}} = \frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_3}$$

$\boxed{\text{HS}}$ Note that since $Y$ is P+Q wrt $A_1, E_1$ $\theta^*_0, \theta^*_1, \cdots, \theta^*_D$, $\theta_0, \theta_1, \cdots, \theta_D$ are all distinct.

So

$$\beta = \frac{1}{\sigma} - 1 = \frac{\theta^*_i - \theta^*_{i+1} + \theta^*_{i+2} - \theta^*_{i+3}}{\theta^*_{i+1} - \theta^*_{i+2}} = \frac{\theta^*_0 - \theta^*_1 + \theta^*_2 - \theta^*_3}{\theta^*_1 - \theta^*_2}$$

We have the assertion.

Given the intersection numbers of a distance-regular graph $\Gamma$. The following 2 lemmas give an efficient method to determine if $\Gamma$ is $Q$-polynomial wrt some primitive idempotent.

LEMMA 46. Let $\Gamma$ be a distance-regular graph of diameter $D \geq 1$.

Pick $\theta, \theta_0^*, \theta_1^*, \ldots, \theta_D^* \in \mathbb{R}$ s.t. $\theta_0^* \neq 0$ and set

$$E = \frac{1}{|X|} \sum_{i=0}^{D} \theta_i^* A_i$$

(i) The following are equivalent

(ia) $\theta$ is an eigenvalue of $\Gamma$, and $E$ is a corresponding primitive idempotent.

(ib)
$$\begin{pmatrix} a_0 & b_0 & & & & 0 \\ c_1 & a_1 & b_1 & & & \\ & c_2 & a_2 & b_2 & & \\ & & & \ddots & & \\ 0 & & & c_{D-1} & a_{D-1} & b_{D-1} \\ & & & & c_D & a_D \end{pmatrix} \begin{pmatrix} \theta_0^* \\ \theta_1^* \\ \vdots \\ \\ \theta_D^* \end{pmatrix} = \theta \begin{pmatrix} \theta_0^* \\ \theta_1^* \\ \vdots \\ \\ \theta_D^* \end{pmatrix}$$

and $\theta_0^* = \operatorname{rank} E$.

(ii) Suppose (ia) (ib) hold.

Then $\dfrac{\theta_1^*}{\theta_0^*}, \ldots, \dfrac{\theta_D^*}{\theta_0^*}$ can be computed from $\theta$

using $$\frac{\theta_i^*}{\theta_0^*} = \frac{p_i(\theta)}{k b_1 \cdots b_{i-1}} \qquad (1 \leq i \leq D)$$

where
$$p_0 = 1$$
$$p_1(\lambda) = \lambda$$
$$\lambda p_i(\lambda) = p_{i+1}(\lambda) + a_i p_i(\lambda) + b_{i-1} c_i p_{i-1}(\lambda)$$
$$(0 \leq i \leq D)$$

Proof.

(i) (ia)

$$\Leftrightarrow \quad (A-\theta I)E = 0 \quad \text{and} \quad E^2 = E$$

$$\Leftrightarrow \quad 0 = \sum_{i=0}^{D} (A-\theta I)\,\theta_i^* A_i \quad \text{and} \quad \operatorname{rank} E = \operatorname{trace} E = \theta_0^*$$

$$= \sum_{i=0}^{D} \theta_i^* \left( c_{i+1} A_{i+1} + a_i A_i + b_{i-1} A_{i-1} - \theta A_i \right)$$

$$= \sum_{j=0}^{D} A_j \left( c_j \theta_{j-1}^* + a_j \theta_j^* + b_j \theta_{j+1}^* - \theta\,\theta_j^* \right)$$

[ $\boxed{HS}$ → is clear ← By the first condition $AE = \theta E$. So $E$ is a scalar multiple of the primitive idempotent corresponding to $\theta$. Hence $\operatorname{rank} E = \operatorname{trace} E$ implies $E$ is the primitive idempotent. ]

$$\Leftrightarrow \quad c_j \theta_{j-1}^* + a_j \theta_j^* + b_j \theta_{j+1}^* = \theta\,\theta_j^* \quad (0 \le j \le D)$$

and $\operatorname{rank} E = \theta_0^*$

$\Leftrightarrow$ (ib)

(ii) We prove by induction on $i$.

$i=0$ : trivial.

$i=1$ : set $j=0$ above $c_0 = 0$ $a_0 = 0$, $b_0 = k$

We have

$$k\,\theta_1^* = \theta\,\theta_0^*$$

So

$$\frac{\theta_1^*}{\theta_0^*} = \frac{\theta}{k} = \frac{p_1(\theta)}{k}.$$

$i \ge 2$ : set $j = i-1$ above.

we have

$$c_{i-1}\theta_{i-2}^* + a_{i-1}\theta_{i-1}^* + b_{i-1}\theta_i^* = \theta\,\theta_{i-1}^*$$

So

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$$\frac{\theta_i^*}{\theta_0^*} = \frac{\theta\,\theta_{i-1}^* - a_{i-1}\,\theta_{i-1}^* - c_{i-1}\,\theta_{i-2}^*}{b_{i-1}\,\theta_0^*}$$

$$= \left( (\theta - a_{i-1}) \frac{\theta_{i-1}^*}{\theta_0^*} - c_{i-1} \frac{\theta_{i-2}^*}{\theta_0^*} \right) \frac{1}{b_{i-1}}$$

$$= \left( (\theta - a_{i-1}) \frac{p_{i-1}(\theta)}{k b_1 \cdots b_{i-2}} - c_{i-1} \frac{p_{i-2}(\theta)}{k b_1 \cdots b_{i-3}} \right) \frac{1}{b_{i-1}}$$

$$= \frac{p_i(\theta)}{k b_1 \cdots b_{i-2} b_{i-1}},$$

as desired.

# Lecture 27 Fri. April 2, 1993

Theorem 47 Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$.
Let $\theta$ denote an eigenvalue of $\Gamma$ with associated primitive idempotent

$$E = \frac{1}{|X|}\sum_{i=0}^{D} \theta_i^* A_i .$$

Then the following are equivalent.

(i) $\Gamma$ is Q-polynomial w.r.t. $E$.

(ii) $\theta_0^* \neq \theta_h^* \quad \forall h \in \{1, 2, \cdots, D\}$ and

$$c_i\left(\theta_2^* - \theta_i^* - \frac{(\theta_1^* - \theta_{i-1}^*)^2}{\theta_0^* - \theta_i^*}\right) + b_{i-1}\left(\theta_2^* - \theta_{i-1}^* - \frac{(\theta_1^* - \theta_i^*)^2}{\theta_0^* - \theta_{i-1}^*}\right)$$
$$= (k-\theta)(\theta_1^* + \theta_2^* - \theta_{i-1}^* - \theta_i^*) - (\theta+1)(\theta_0^* - \theta_2^*) \quad -(1)$$
$$(3 \leq i \leq D)$$

(iii) $\theta_0^* \neq \theta_h^* \quad \forall h \in \{1, 2, \cdots, D\}$ and (1) holds for $i = 3$.

(Note (1) is trivial for $i = 1, 2$).

$\boxed{\text{HS}}$ $i=1$ : 
$$\text{LHS} = \left(\theta_2^* - \theta_1^* - \frac{(\theta_1^* - \theta_0^*)^2}{\theta_0^* - \theta_1^*}\right) + k(\theta_2^* - \theta_0^*)$$
$$= \theta_2^* - \cancel{\theta_1^*} - \theta_0^* + \cancel{\theta_1^*} + k(\theta_2^* - \theta_0^*)$$
$$= (k+1)(\theta_2^* - \theta_0^*)$$

$$\text{RHS} = (k-\theta)(\cancel{\theta_1^*} + \theta_2^* - \theta_0^* - \cancel{\theta_1^*}) - (\theta+1)(\theta_0^* - \theta_2^*)$$
$$= (k+1)(\theta_2^* - \theta_0^*)$$

$i=2$ : 
$$\text{LHS} = b_1\left(\theta_2^* - \theta_1^* - \frac{(\theta_1^* - \theta_2^*)^2}{\theta_0^* - \theta_1^*}\right)$$
$$= b_1 \frac{(\theta_2^* - \theta_1^*)(\theta_0^* - \cancel{\theta_1^*} - \theta_2^* + \cancel{\theta_1^*})}{\theta_0^* - \theta_1^*}$$

$$= b_1 \frac{(\theta_2^* - \theta_1^*)(\theta_0^* - \theta_2^*)}{\theta_0^* - \theta_1^*}$$

$$\text{RHS} = -(\theta+1)(\theta_0^* - \theta_2^*)$$

$$\text{LHS} = \text{RHS} \Leftrightarrow b_1 \frac{\theta_2^* - \theta_1^*}{\theta_0^* - \theta_1^*} + (\theta+1) = 0$$

$$\Leftrightarrow b_1(\theta_2^* - \theta_1^*) + (\theta+1)(\theta_0^* - \theta_1^*) = 0$$

On the other hand

$$\left.\begin{aligned} b_1\theta_2^* + a_1\theta_1^* + c_1\theta_0^* &= \theta\theta_1^* \\ b_1\theta_1^* + a_1\theta_1^* + c_1\theta_1^* &= k\theta_1^* \\ (\because \quad \theta\theta_0^* &= k\theta_1^*) \end{aligned}\right] \to \text{See Lec26-5}$$

$$\therefore b_1(\theta_2^* - \theta_1^*) + (\theta_0^* - \theta_1^*) = \theta(\theta_1^* - \theta_0^*)$$

Proof. Immediate from the proof of Thm 2.1 in "a new inequality for distance-regular graphs" and Thm 44.

Note: Suppose (i) — (iii) hold.
In particular, $\theta_0^*, \theta_1^*, \cdots, \theta_D^*$ are distinct.
Then.

$$c_i + a_i + b_i = k \qquad (0 \le i \le D)$$

$$c_i\theta_{i-1}^* + a_i\theta_i^* + b_i\theta_{i+1}^* = \theta\theta_i^* \qquad (0 \le i \le D)$$

$$\frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} \quad \text{is independent of } i \quad (0 \le i \le D-3)$$

$$c_i\left(\theta_2^* - \theta_i^* - \frac{(\theta_1^* - \theta_{i-1}^*)^2}{\theta_0^* - \theta_i^*}\right) + b_{i-1}\left(\theta_2^* - \theta_{i-1}^* - \frac{(\theta_1^* - \theta_i^*)^2}{\theta_0^* - \theta_{i-1}^*}\right)$$

$$= (k-\theta)(\theta_1^* + \theta_2^* - \theta_{i-1}^* - \theta_i^*) - (\theta+1)(\theta_0^* - \theta_2^*)$$

Furthermore, we can solve for $c_1, \cdots, c_D$, $a_1, \cdots a_D$, $b_0, b_1, \cdots, b_{D-1}$ in terms of 5 free parameters

In general, we can take the 5 parameters to be $D$, $q$, $s^*$, $r_1$, $r_2$ and get

$$b_i = \frac{h(1-q^{i-D})(1-s^*q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})} \qquad (0 \le i \le D)$$

$$c_i = \frac{h(1-q^i)(1-s^*q^{D+i+1})(r_1-s^*q^i)(r_2-s^*q^i)}{s^*q^D(1-s^*q^{2i})(1-s^*q^{2i+1})} \qquad (0 \le i \le D)$$

$$a_i = b_0 - c_i - b_i \qquad (0 \le i \le D)$$

where $h$ variable is chosen so

$$c_1 = 1.$$

(Must also consider limiting cases $h \to 0$
$s^* \to 0$, $q \to \mp 1$.
See Thm 2.1 in
"The subconstituent algebra of an association scheme,
Journal of Algebraic Combinatorics
Part I, Vol 1 (1992), 363–388; Part II, Vol 2 (1993), 73–103;
Part III, Vol 2 (1993), 177–210

DEF. Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$.
Choose: $q \in \mathbb{R} \setminus \{0, -1\}$. set

$$\begin{bmatrix} i \\ 1 \end{bmatrix} = 1 + q + \cdots + q^{i-1} = \begin{cases} \dfrac{q^i - 1}{q - 1} & q \neq 1 \\ i & q = 1 \end{cases}$$

DEF $\Gamma$ has <u>classical parameters</u> if

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left( 1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \qquad (1)$$

$$(0 \leq i \leq D)$$

$$b_i = \left( \begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left( \sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \qquad (2)$$

for some $\sigma, \alpha \in \mathbb{R}$.
( This happens for essentially all known $\infty$ families of distance regular graphs with unbounded diameter, and is essentially equivalent to $s^* = 0$ )

LEMMA 48 With above notation, suppose (1), (2) hold. Then

(i) $\theta := \dfrac{b_1}{q} - 1$ is an eigenvalue of $\Gamma$ with $\theta \neq k$,

(ii) Let $E = \dfrac{1}{|X|} \sum_{i=0}^{D} \theta_i^* A_i$ be associated primitive idempotent.

Then $$\frac{\theta_i^*}{\theta_0^*} = 1 + \left( \frac{\theta}{k} - 1 \right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \qquad (0 \leq i \leq D)$$

In particular $\theta_i^* \neq \theta_0^*$ $\forall i \in \{1, 2, \ldots, D\}$

(iii) $\Gamma$ is $Q$-polynomial w.r.t. $E$.

Proof

(i), (ii) Need to check

$$c_i\theta^*_{i-1} + a_i\theta^*_i + b_i\theta^*_{i+1} = \theta\theta^*_i \qquad (0\le i\le D)$$

where $a_i := k - c_i - b_i \qquad (0\le i\le D)$

(equivalently : check

$$c_i(\theta^*_{i-1}-\theta^*_i) - b_i(\theta^*_i - \theta^*_{i+1}) = (\theta - k)\theta^*_i \qquad (0\le i\le D) \quad — \star$$

where $c_i,\ b_i,\ \theta^*_i,\ \theta$ are as given

$\boxed{\text{HS}}$ $\theta = \frac{b_1}{q}+1 \qquad \frac{\theta^*_i}{\theta^*_0} = 1+\left(\frac{\theta}{k}-1\right)\left[{i \atop 1}\right] q^{1-i} \qquad b_0 = \left[{D \atop 1}\right]\sigma = k$

$i=0 \quad \frac{\theta^*_1}{\theta^*_0} = \frac{\theta}{k} \qquad -k\left(1-\frac{\theta^*_1}{\theta^*_0}\right) = -k\left(1-\frac{\theta}{k}\right) = \theta - k. \quad \star_0$ : OK

$$\frac{\theta^*_{i-1}-\theta^*_i}{\theta^*_0} = \left(\frac{\theta}{k}-1\right)\left(\left[{i-1 \atop 1}\right]q^{2-i} - \left[{i \atop 1}\right]q^{1-i}\right) = -\left(\frac{\theta}{k}-1\right)q^{1-i}$$

$$\theta - k = \left(\left(\left[{D \atop 1}\right]-1\right)(\sigma-\alpha)\right)/q - 1 - \left[{D \atop 1}\right]\sigma = \left[{D-1 \atop 1}\right](\sigma-\alpha) - 1 - \left[{D \atop 1}\right]\sigma$$

$$\left(c_i(\theta^*_{i-1}-\theta^*_i) - b_i(\theta^*_i-\theta^*_{i+1}) - (\theta-k)\theta^*_i\right)/\theta^*_0$$

$$= -\left[{i \atop 1}\right]\left(1+\alpha\left[{i-1 \atop 1}\right]\right)\left(\frac{\theta}{k}-1\right)q^{1-i} + \left(\left[{D \atop 1}\right]-\left[{i \atop 1}\right]\right)\left(\sigma-\alpha\left[{i \atop 1}\right]\right)\left(\frac{\theta}{k}-1\right)q^{-i}$$

$$-(\theta-k)\left(1+\left(\frac{\theta}{k}-1\right)\left[{i \atop 1}\right]q^{1-i}\right)$$

$$= \left(\frac{\theta}{k}-1\right)\Big\{-\left[{i \atop 1}\right]\left(1+\alpha\left[{i-1 \atop 1}\right]\right)q^{1-i} + \left[{D-i \atop 1}\right]\left(\sigma-\alpha\left[{i \atop 1}\right]\right)$$

$$-\left(\left[{D \atop 1}\right]\sigma + \left(\left[{D-1 \atop 1}\right](\sigma-\alpha) - 1 - \left[{D \atop 1}\right]\sigma\right)\left[{i \atop 1}\right]q^{1-i}\right)\Big\}$$

$$= \left(\frac{\theta}{k}-1\right)\Big\{-\cancel{\left[{i \atop 1}\right]q^{1-i}} - \alpha\Big(\overbrace{\left[{i \atop 1}\right]\left[{i-1 \atop 1}\right]q^{1-i}}^{(1+\cdots+q^{i-2})q^{1-i}} + \overbrace{\left[{D-i \atop 1}\right]\left[{i \atop 1}\right]}^{1+\cdots+q^{D-i-1}} - \overbrace{\left[{D-1 \atop 1}\right]\left[{i \atop 1}\right]q^{1-i}}^{(1+\cdots+q^{D-2})q^{1-i}}\Big)$$

$$+\sigma\Big(\underbrace{\overbrace{\left[{D-i \atop 1}\right]-\left[{D \atop 1}\right]}^{q^{D-i}+\cdots+q^{D}} - \left[{D-1 \atop 1}\right]\left[{i \atop 1}\right]q^{1-i} + \left[{D \atop 1}\right]\left[{i \atop 1}\right]q^{1-i}}_{0}\Big) + \cancel{\left[{i \atop 1}\right]q^{1-i}}\Big)$$

Check $\theta \neq k$: Suppose $\theta = k$.

Then $\dfrac{b_1}{q} - 1 = k$ and $q > 0$.

By (1), (2)

$$q c_i - b_i - q(q c_{i-1} - b_{i-1}) = (k - \theta) q \qquad (1 \leq i \leq D)$$
$$= 0$$

$\boxed{\text{HS}}$ With the notation of Lemma 48, we have the above equality in general.

$$q c_i - b_i - q(q c_{i-1} - b_{i-1})$$
$$= q\genfrac{[}{]}{0pt}{}{i}{1}\left(1+\alpha\genfrac{[}{]}{0pt}{}{i-1}{1}\right) - \left(\genfrac{[}{]}{0pt}{}{D}{1} - \genfrac{[}{]}{0pt}{}{i}{1}\right)\left(\sigma - \alpha\genfrac{[}{]}{0pt}{}{i}{1}\right) - q\left(q\genfrac{[}{]}{0pt}{}{i-1}{1}\left(1+\alpha\genfrac{[}{]}{0pt}{}{i-2}{1}\right) - \left(\genfrac{[}{]}{0pt}{}{D}{1} - \genfrac{[}{]}{0pt}{}{i-1}{1}\right)\left(\sigma - \alpha\genfrac{[}{]}{0pt}{}{i-1}{1}\right)\right)$$
$$= \left\{q\genfrac{[}{]}{0pt}{}{i}{1} - q^2\genfrac{[}{]}{0pt}{}{i-1}{1}\right\} + \alpha\left\{q\genfrac{[}{]}{0pt}{}{i}{1}\genfrac{[}{]}{0pt}{}{i-1}{1} + \genfrac{[}{]}{0pt}{}{D}{1}\genfrac{[}{]}{0pt}{}{i}{1} - \genfrac{[}{]}{0pt}{}{i}{1}\genfrac{[}{]}{0pt}{}{i}{1} - q^2\genfrac{[}{]}{0pt}{}{i-1}{1}\genfrac{[}{]}{0pt}{}{i-2}{1} - q\genfrac{[}{]}{0pt}{}{D}{1}\genfrac{[}{]}{0pt}{}{i-1}{1} + q\genfrac{[}{]}{0pt}{}{i-1}{1}\genfrac{[}{]}{0pt}{}{i-1}{1}\right\}$$
$$+ \sigma\left\{-\genfrac{[}{]}{0pt}{}{D}{1} + \genfrac{[}{]}{0pt}{}{i}{1} + q\genfrac{[}{]}{0pt}{}{D}{1} - q\genfrac{[}{]}{0pt}{}{i-1}{1}\right\}$$
$$= q + \alpha\left\{-\genfrac{[}{]}{0pt}{}{i}{1} + \genfrac{[}{]}{0pt}{}{D}{1} + q\genfrac{[}{]}{0pt}{}{i-1}{1}\right\} + \sigma\left\{q^D - 1 + 1\right\}$$
$$= q\left(1 + \genfrac{[}{]}{0pt}{}{D-1}{1}\alpha + q^{D-1}\sigma\right)$$
$$= q\left(\genfrac{[}{]}{0pt}{}{D}{1}\sigma - \genfrac{[}{]}{0pt}{}{D-1}{1}\sigma + \genfrac{[}{]}{0pt}{}{D-1}{1}\alpha + 1\right) = q\left(k - \frac{\genfrac{[}{]}{0pt}{}{D}{1} - 1}{q}(\sigma - \alpha) + 1\right)$$
$$= q(k - \theta).$$

Hence

$$q c_i - b_i = q(q c_{i-1} - b_{i-1}) \qquad (1 \leq i \leq D)$$
$$= q^i (q c_0 - b_0)$$
$$= -q^i k$$

If $i = D$, $\quad q c_D = -q^D k$

$c_D = -q^{D-1} k < 0$, a contradiction

(iii) Check the equation (ii) of Thm 47 holds for $i=3$.

$\boxed{\text{HS}}$ $\quad \theta_0^* \neq \theta_h^* \quad \forall h \in \{1,2,\cdots,D\}$ and

$$c_3\left(\theta_2^*-\theta_3^*-\frac{(\theta_1^*-\theta_2^*)^2}{\theta_0^*-\theta_3^*}\right)-b_2\frac{(\theta_1^*-\theta_3^*)^2}{\theta_0^*-\theta_2^*}$$

$$=(k-\theta)(\theta_1^*-\theta_3^*)-(\theta+1)(\theta_0^*-\theta_2^*)$$

$$\frac{\text{LHS}}{\theta_0^*}=\left[{3\atop 1}\right]\left(1+\alpha\left[{2\atop 1}\right]\right)\left\{\left(1-\frac{\theta}{k}\right)\left(q^{-2}-\frac{q^{-2}}{\left[{3\atop 1}\right]q^{-2}}\right)\right\}$$

$$-\left(\underbrace{\left[{D\atop 1}\right]-\left[{2\atop 1}\right]}_{q^2\left[{D-2\atop 1}\right]}\right)\left(\sigma-\alpha\left[{2\atop 1}\right]\right)\left(1-\frac{\theta}{k}\right)\frac{\left(\left[{3\atop 1}\right]\cancel{q^{-3}}-1\right)^2}{\cancel{\left[{2\atop 1}\right]q^{-1}}} \qquad \left[{2\atop 1}\right]q^{-3} \qquad \frac{\left[{3\atop 1}\right]-q^2}{\left[{2\atop 1}\right]} \quad 1+q+q^2$$

$$=\left(1-\frac{\theta}{k}\right)\left(\left(1+\alpha\left[{2\atop 1}\right]\right)\left(q^{-2}\left[{2\atop 1}\right]\right)-\left(\underbrace{\left[{D\atop 1}\right]-\left[{2\atop 1}\right]}_{q^2\left[{D-2\atop 1}\right]}\right)\left(\sigma-\alpha\left[{2\atop 1}\right]\right)\left[{2\atop 1}\right]q^{-3}\right)$$

$$=\left(1-\frac{\theta}{k}\right)\left(q^{-2}\left[{2\atop 1}\right]+\alpha\underbrace{\left[q^{-2}\left[{2\atop 1}\right]\left[{2\atop 1}\right]+q^{-1}\left[{2\atop 1}\right]\left[{2\atop 1}\right]\left[{D-2\atop 1}\right]\right)}_{\alpha q^{-2}\left[{2\atop 1}\right]\left[{2\atop 1}\right]\left[{D-1\atop 1}\right]}-q^{-1}\left[{2\atop 1}\right]\left[{D-2\atop 1}\right]\sigma\right)$$

$$\frac{\text{RHS}}{\theta_0^*}=\left(\left[{D\atop 1}\right]\sigma-\underbrace{\left[{D-1\atop 1}\right]}_{q^{D-1}\sigma}(\sigma-\alpha)+1\right)\left(1-\frac{\theta}{k}\right)\left(\underbrace{\left[{3\atop 1}\right]q^{-2}-1}_{q^{-2}\left[{2\atop 1}\right]}\right)$$

$$-\left(\left[{D-1\atop 1}\right](\sigma-\alpha)\right)\left(1-\frac{\theta}{k}\right)\left[{2\atop 1}\right]q^{-1}$$

$$=\left(1-\frac{\theta}{k}\right)\left(q^{-2}\left[{2\atop 1}\right]+\left[{2\atop 1}\right]q^{-1}\sigma\left(q^{D-2}-\left[{D-1\atop 1}\right]\right)\right)+\left[{2\atop 1}\right]q^{-2}\alpha\left(\left[{D-1\atop 1}\right]+q\left[{D-1\atop 1}\right]\right)$$

$$=\left(1-\frac{\theta}{k}\right)\left(q^{-2}\left[{2\atop 1}\right]-\sigma q^{-1}\left[{2\atop 1}\right]\left[{D-2\atop 1}\right]+\alpha q^{-2}\left[{2\atop 1}\right]\left[{2\atop 1}\right]\left[{D-1\atop 1}\right]\right)$$

Examples. Q-polynomial distance-regular graphs with classical parameters

D-cube

$$c_i = i$$
$$b_i = D - i$$

has classical parameters. $q = 1,\ \alpha = 0,\ \sigma = 1$

Johnson graph $J(D, N)$ $(N \geq 2D)$

$$c_i = i^2$$
$$b_i = (D-i)(N-D-i)$$

has classical parameters $q = 1,\ \alpha = 1,\ \sigma = N - D.$

$q$-analogue of Johnson graph $J_q(D, N)$ $(N \geq 2D)$

$$c_i = \left(\frac{q^i - 1}{q - 1}\right)^2 = \begin{bmatrix} i \\ 1 \end{bmatrix}_q^2$$

$$b_i = \frac{q(q^D - q^i)(q^{N-D} - q^i)}{(q-1)^2} =$$

has classical parameters $q$ as above $\alpha = q$

$$\sigma = \left(\frac{q^{N-D+1} - 1}{q - 1}\right) - 1 = \begin{bmatrix} N-D+1 \\ 1 \end{bmatrix}_q - 1$$

$\boxed{\text{HS}}$ $$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\begin{bmatrix} N-D+1 \\ 1 \end{bmatrix} - 1 - q\begin{bmatrix} i \\ 1 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\begin{bmatrix} N-D+1 \\ 1 \end{bmatrix} - \begin{bmatrix} i+1 \\ 1 \end{bmatrix}\right)$$

$$= \frac{q(q^D - q^i)(q^{N-D} - q^i)}{(q-1)^2}$$

# Lecture 28 Mon. April 5, 1993

LEMMA 49 Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$ with standard module $V$.
Suppose $\Gamma$ is $Q$-polynomial w.r.t. a primitive idempotent $E_1$. Pick $x \in X$.
Then $E_1 V = \operatorname{Span}\{ E_i y \mid \partial(x, y) \leq 2 \}$

In particular,

$$\dim E_1 V \leq 1 + k_1 + k_2$$

Proof.

Let $\Delta = \{ E_i y \mid \partial(x, y) \leq 2 \}$.

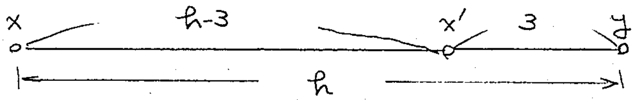
$\supseteq$ : clear

$\subseteq$ : Pick $y \in X$. Show $E_1 y \in \operatorname{Span} \Delta$

Induction on $h := \partial(x, y)$

<u>Case</u> $h \leq 2$ : $Ey \in \operatorname{Span} \Delta$ follows from construction

<u>Case</u> $h \geq 3$ : Pick $x' \in X$ s.t.
$\partial(x, x') = h-3$, $\partial(x', y) = 3$.




By Theorem 44

$$\underbrace{\sum_{\substack{z \in X \\ \partial(x', z) = 1 \ \ \partial(y, z) = 2}} E_1 \hat{z}}_{f} - \underbrace{\sum_{\substack{z' \in X \\ \partial(x', z') = 2 \ \ \partial(y, z') = 1}} E_1 \hat{z}'}_{g} = r_{12}^3 (E_1 \hat{x} - E_1 \hat{y})$$

$$r_{12}^3 = \frac{c_3 (\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_3^*} \neq 0$$

So $E_1 \hat{y} \in \operatorname{Span}\{ f, g, E_1 \hat{x} \}$

Observe : each $z$ in the $f$-sum satisfies
$\partial(x, z) = h-2$
So by induction hypothesis
$E_1\hat{z} \in \operatorname{Span}\Delta$ or $f \in \operatorname{Span}\Delta$.

Observe : each $z'$ in the $g$-sum satisfies
$\partial(x, z') = h-1$
So by induction hypothesis
$E_1\hat{z}' \in \operatorname{Span}\Delta$ or $g \in \operatorname{Span}\Delta$.

Also $\partial(x\, x') = h-3$ implies
$E_1\hat{x}' \in \operatorname{Span}\Delta$

Therefore $E_1\hat{y} \in \operatorname{Span}\Delta$.

Note : Let $\Gamma$, $E_1$, $x$ be as in Lemma 49.
Assume $D \geq 4$.
Observe there are many linear dependences among
$$\{ E\hat{y} \mid y \in \Delta \}$$
where
$$\Delta = \{ y \in X \mid \partial(x, y) \leq 2 \}.$$

Take any $y \in X$ s.t. $\partial(x, y) \geq 4$.
More than one choice for $x'$ in the proof of Lemma 49 implies
"more than one way to put $E_1\hat{y} \in \operatorname{Span} E_1\Delta$".

Open problem : (i) Give a precise description of the linear dependences among
$$\{ E_1\hat{y} \mid y \in \Delta \}$$

(ii) Find a subset $\Delta' \subseteq \Delta$ such that
$$\{ E_1\hat{y} \mid y \in \Delta' \}$$
is a basis for $E_1V$,
( or find some other 'nice' basis for $E_1V$ )

<u>Conjecture</u> Let $\Gamma$, $E_1$, $x$ be as in Lemma 49.
Set $\tilde{X} = \{ y \in X \mid \partial(x, y) \leq 2 \}$.

$\tilde{\partial}$ = the restriction of the distance function $\partial$ to $\tilde{X}$

Then $\Gamma$ is determined by $\tilde{X}$, $\tilde{\partial}$.

(There should be some canonical way to reconstruct $\Gamma$ from $\tilde{X}$, $\tilde{\partial}$.)

# Lecture 29 Wed. April 7, 1995

## Introduction to Theorem 50

Let $\Gamma = (X, E)$ be distance-regular with diameter $D \geq 3$.
Assume $\Gamma$ is Q-polynomial wrt $E_1$.
Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $A_i^* \equiv A_i^*(x)$, $A^* \equiv A_1^*$

We know

$$E_i^* A_h E_j^* = 0 \quad \leftrightarrow \quad p_{ij}^h = 0 \qquad (0 \leq h, i, j \leq D)$$
$$E_i A_h^* A_j = 0 \quad \leftrightarrow \quad q_{ij}^h = 0$$

Also

$$h < |i-j| \quad \rightarrow \quad p_{ij}^h = 0 \quad q_{ij}^h = 0 \qquad (0 \leq h, i, j \leq D)$$
$$h = |i-j| \quad \rightarrow \quad p_{ij}^h \neq 0, \quad q_{ij}^h \neq 0$$

Since $A_h$ (resp. $A_h^*$) is a polynomial of degree exactly $h$ in $A$ (resp. $A^*$), it follows

$$E_i^* A^h E_j^*, \quad E_i A^{*h} E_j = \begin{cases} = 0 & \text{if } h < |i-j| \\ \neq 0 & \text{if } h = |i-j| \end{cases}$$
$$(0 \leq h, i, j) \leq D.$$

We saw $\exists\, \beta, \gamma, \delta \in \mathbb{R}$ s.t.

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*]$$

In fact, $\exists\, \beta, \gamma, \delta \in \mathbb{R}$ s.t

$$0 = [A, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma(A^*A + AA^*) - \delta A]$$

as well as we will now show.

Let $K$ denote any field. Let $V$ denote any vector space $/K$ of finite positive dimension. Let $\mathrm{End}_K(V)$ denote the $K$-algebra of all $K$-linear transformations $V \longrightarrow V$

Theorem 50 Given semi-simple elements $A, A^* \in \mathrm{End}_K(V)$ suppose

$$E_i (A^*)^h E_j = \begin{cases} = 0 & \text{if } h < |i-j| \\ \neq 0 & \text{if } h = |i-j| \end{cases} \quad (0 \le h, i, j \le D) \quad (1)$$

$$E_i^* A^h E_j^* = \begin{cases} = 0 & \text{if } h < |i-j| \\ \neq 0 & \text{if } h = |i-j| \end{cases} \quad (0 \le h, i, j \le R) \quad (2)$$

for some ordering $E_0, E_1, \cdots, E_D$ of the primitive idempotents for $A$,

and some ordering $E_0^*, E_1^*, \cdots, E_R^*$ of the primitive idempotents for $A^*$.

Then

(i) $R = D$

(ii) $\exists\, \beta, \gamma, \gamma^*, \delta, \delta^* \in K$ s.t.

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*] \quad (3)$$

$$= A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A) \quad (4)$$

$$0 = [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A] \quad (5)$$

$$= A^{*3}A - AA^{*3} - (\beta+1)(A^{*2}AA^* - A^*AA^{*2}) - \gamma^*(A^{*2}A - AA^{*2}) - \delta^*(A^*A - AA^*) \quad (6)$$

(iii) Let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) associated with $E_i$ (resp $E_i^*$),

Then

$$\beta = \frac{\theta_i - \theta_{i+1} + \theta_{i+2} - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}} \quad (0 \le i \le D-3) \quad (7)$$

$$= \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} \quad (0 \le i \le D-3) \quad (8)$$

$$\gamma = \theta_i - \beta\theta_{i+1} + \theta_{i+2} \qquad (0 \le i \le D-2) \qquad (9)$$

$$\gamma^* = \theta_i^* - \beta\theta_{i+1}^* + \theta_{i+2}^* \qquad (0 \le i \le D-2) \qquad (10)$$

$$\delta = \theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) \qquad (0 \le i \le D-1) \qquad (11)$$

$$\delta^* = \theta_i^{*2} - \beta\theta_i^*\theta_{i+1}^* + \theta_{i+1}^{*2} - \gamma^*(\theta_i^* + \theta_{i+1}^*) \qquad (0 \le i \le D-1) \qquad (12)$$

In particular, $\beta, \gamma, \gamma^*, \delta, \delta^*$ are uniquely determined by $A$, $A^*$ and the above orderings of their primitive idempotents, whenever $D \ge 3$.

Proof of (i).

By symmetry, it suffices to show $D \ge R$.
Suppose $R > D$.
Since $A$ is semisimple with exactly $D+1$ distinct eigenvalues, the minimal polynomial of $A$ has degree $D+1$.
Since $R \ge D+1$,

$$A^R \in \mathrm{Span}\{ A^j \mid 0 \le j \le D \}.$$

Multiplying each term on the left by $E_R^*$ and on the right by $E_0^*$, we find

$$E_R^* A^R E_0^* \in \mathrm{Span}\{ E_R^* A^j E_0^* \mid 0 \le j \le D \} \quad - (13)$$

But by (2), the left side of (13) is nonzero and the right side of (13) is 0, a contradiction.
Hence $D \ge R$.

Proof of (ii), (iii)

Recalling the definitions, we have

$$A = \sum_{i=0}^{D} \theta_i E_i$$

$$A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$$

$$AE_i = E_i A = \theta_i E_i \qquad (0 \le i \le D)$$

$$A^* E_i^* = E_i^* A^* = \theta_i^* E_i^* \qquad (0 \le i \le D)$$

Claim 1 For all integers $i, j, k, l$ $(0 \le i, j, k, l \le D)$ such that $j + k \le i - l$

$$E_i^* A^j A^* A^k E_l^* \tag{14}$$

$$= \begin{cases} \theta_{l+k}^* E_i^* A^{j+k} E_l^* & \text{if } j+k = i-l \\ 0 & \text{if } j+k < i-l \end{cases}$$

Proof of Claim 1

The product (14) equals

$$E_i^* A^j \left( \sum_{h=0}^{D} \theta_h^* E_h^* \right) A^k E_l^*$$

$$= \sum_{h=0}^{D} \theta_h^* E_i^* A^j E_h^* A^k E_l^*$$

Now pick any $h$ $(0 \le h \le D)$, where

$$E_i^* A^j E_h^* A^k E_l^* \ne 0.$$

Then by (2) $j \ge |i-h|$, otherwise

$$E_i^* A^j E_h^* = 0$$

and by (1) $k \ge |h - l|$, otherwise

$$E_h^* A^k E_\ell^* = 0$$

Hence

$$\begin{aligned} j + k &\geq |i - h| + |h - \ell| \\ &\geq |i - \ell| \\ &\geq i - \ell \end{aligned}$$

Now if $j + k < i - \ell$, we see there is no such $h$, so (14) holds.

If $j + k = i - \ell$,
$h = \ell + k$ is the only solution, so (14) holds.

[ $(\because)$ $i = j + k + \ell$ $\quad 0 \leq i, j, k, \ell, h \leq D$.
$\therefore i \geq j, k, \ell$. Since $k = |h - \ell|$,
if $h \neq \ell + k$, $\therefore h = \ell - k$ and $j = i - h$
$\ell - h + \cancel{i} - h = \cancel{i} - \ell$ $\quad h = \ell$ $\quad k = 0$ and $h = \ell + k$. ]

This proves claim 1.

Let $M$ denote the subalgebra of $\mathrm{End}_K(V)$ generated by $A$. Observe that $M$ has basis $E_0, \cdots, E_D$ as a vector space $/K$.

Set $\quad L := \mathrm{Span}\{ m A^* n - n A^* m \mid n, m \in M \}$

Claim 2 $\quad \dim L \leq D$.

<u>Proof of Claim 2</u> $\quad E_0, \cdots, E_D$ span $M$, so

$$L = \mathrm{Span}\{ E_i A^* E_j - E_j A^* E_i \mid 0 \leq i < j \leq D \}$$

$$= \mathrm{Span}\{ E_{j-1} A^* E_j - E_j A^* E_{j-1} \mid 1 \leq j \leq D \}$$

by (1). In particular $L$ has a spanning set of order $D$. So Claim 2 holds.

Claim 3 $\{ A^i A^* - A^* A^i \mid 1 \le i \le D \}$ is a basis for L.

Proof of Claim 3 Since $A^i A^* - A^* A^i = A^i A^* I - I A^* A^i$ is contained in L $(1 \le i \le D)$, and since $\dim L \le D$, it suffices to show the given elements are linearly independent.
Suppose they are dependent. Then there exists an integer $i$ $(1 \le i \le D)$ s.t

$$A^i A^* - A^* A^i \in \mathrm{Span}(A^j A^* - A^* A^j \mid 1 \le j < i) \quad -(15)$$

Multiplying each term in (15) on the left by $E_i^*$ and on the right by $E_0^*$, and simplifying using

$$E_i^*(A^\ell A^* - A^* A^\ell) E_0^* = (\theta_0^* - \theta_i^*) E_i^* A^\ell E_0^*,$$

we find

$$E_i^* A^i E_0^* \in \mathrm{Span}(E_i^* A^j E_0^* \mid 1 \le j < i). \quad -(16)$$

But the left side of (16) is non zero, and the right side of (16) equals zero.
A contradiction.

Since $A^2 A^* A - A A^* A^2$ is contained in L, we find by Claim 2,

$$A^2 A^* A - A A^* A^2 = \sum_{i=1}^{D} \alpha_i (A^i A^* - A^* A^i) \quad -(17)$$

for some $\alpha_0, \cdots, \alpha_D \in K$.

<u>Claim 4</u> $\alpha_i = 0 \quad (3 < i \le D)$

<u>Proof Claim 4</u> Suppose not, and set

$$t := \max\{ i \mid 3 < i \le D,\ \alpha_i \ne 0 \}$$

Then by (17) and Claim 1.

$$0 = E_t^*(A^2A^*A - AA^*A^2 - \sum_{i=1}^{D} \alpha_i (A^iA^* - A^*A^i)) E_0^*$$

$$= \alpha_t(\theta_t^* - \theta_0^*) E_t^* A^t E_0^*$$

$$\ne 0$$

( $E_t^*A^2A^*AE_0^* = E_t^*AA^*A^2E_0^* = 0 \quad (\because\ 2+1 < t-0)$

$E_t^*A^iA^*E_0^* = E_t^*A^*A^iE_0^* = 0 \quad (\text{if } i+0 < t)$

$E_t^*A^tA^*E_0^* = \theta_0^* E_t^*A^tE_0^*$

$E_t^*A^*A^tE_0^* = \theta_t^* E_t^*A^tE_0^*$

$\alpha_i = 0$ if $i > t$. )

A contradiction. This proves Claim 4.

<u>Claim 5</u> Suppose $D \ge 3$. Then

$$\alpha_3 = \frac{\theta_{i+1}^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i+3}^*} \quad \text{for all } i. \quad (0 \le i \le D-3) \qquad (18)$$

In particular, $\alpha_3 \ne 0$.

<u>Proof of Claim 5</u>

Fix an integer $i$ $(0 \le i \le D-3)$. Then

By (14), (17)

$$0 = E_{i+3}^*(A^2A^*A - AA^*A^2 - \sum_{j=1}^{3} \alpha_j (A^jA^* - A^*A^j)) E_i^*$$

$$= (\theta_{i+1}^* - \theta_{i+2}^* - \alpha_3(\theta_i^* - \theta_{i+3}^*)) E_{i+3}^* A^3 E_i^*.$$

But $E_{i+3}^*A^3E_i^* \ne 0$ by (2), so (18) holds

This proves Claim 5

Claim 6 Lines (3), (4), (8) hold.

Proof of Claim 6 First suppose $D \geq 3$.

Then by (17), Claims 4, 5

$$A^2A^*A - AA^*A^2 = \alpha_3(A^3A^* - A^*A^3) + \alpha_2(A^2A^* - A^*A^2) + \alpha_1(AA^* - A^*A) \qquad (19)$$

where $\alpha_3 \neq 0$. Hence

$$A^3A^* - A^*A^3 - \frac{1}{\alpha_3}(A^2A^*A - AA^*A^2) - \left(-\frac{\alpha_2}{\alpha_3}\right)(A^2A^* - A^*A^2) - \left(-\frac{\alpha_1}{\alpha_3}\right)(AA^* - A^*A) = 0$$

Now (4) is immediate, where

$$\beta = \frac{1}{\alpha_3} - 1 \qquad (20)$$

$$\gamma = -\frac{\alpha_2}{\alpha_3} \qquad (21)$$

$$\delta = -\frac{\alpha_1}{\alpha_3} \qquad (22)$$

The line (3) follows from the definition of [ , ].
The line (8) is immediate from (18) and (20).

Now suppose $D < 3$. Then the line (8) is vacuously true, so consider (4).
Let $\alpha_3$ denote any non zero element of $K$.
Then $A^2A^* - A^*A^2$, $AA^* - A^*A$ certainly span $L$ by Claim 3.
So (19) holds for appropriate $\alpha_1, \alpha_2 \in K$.
Now (4) holds where $\beta, \gamma, \delta$ are given by (20) – (22).

Claim 7 Lines (7), (9), (11) hold.

Proof of Claim 7 Pick an integer $i$ $(0 \le i \le D-1)$.
By (4) we have

$$0 = E_i\left(A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)\right)E_{i+1}$$

$$= E_iA^*E_{i+1}\left(\theta_i^3 - \theta_{i+1}^3 - (\beta+1)(\theta_i^2\theta_{i+1} - \theta_i\theta_{i+1}^2) - \gamma(\theta_i^2 - \theta_{i+1}^2) - \delta(\theta_i - \theta_{i+1})\right)$$

$$= E_iA^*E_{i+1}\,(\theta_i - \theta_{i+1})\left(\theta_i^2 + \theta_i\theta_{i+1} + \theta_{i+1}^2 - (\beta+1)\theta_i\theta_{i+1} - \gamma(\theta_i + \theta_{i+1}) - \delta\right)$$

$$= E_iA^*E_{i+1}\,(\theta_i - \theta_{i+1})\left(\theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \delta\right)$$

But $E_iA^*E_{i+1} \ne 0$ by (1), and of course $\theta_i \ne \theta_{i+1}$, so

$$0 = \theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \delta.$$

This gives (11).

To obtain (9), Pick any integer $i$ $(0 \le i \le D-2)$.
Then by (11),

$$0 = \theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \delta$$
$$\quad - (\theta_{i+1}^2 - \beta\theta_{i+1}\theta_{i+2} + \theta_{i+2}^2 - \gamma(\theta_{i+1} + \theta_{i+2}) - \delta)$$

$$= \theta_i^2 - \beta\theta_i\theta_{i+1} - \gamma\theta_i + \beta\theta_{i+1}\theta_{i+2} - \theta_{i+2}^2 + \gamma\theta_{i+2}$$

$$= (\theta_i - \theta_{i+2})(\theta_i - \beta\theta_{i+1} + \theta_{i+2} - \gamma).$$

So $\quad 0 = \theta_i - \beta\theta_{i+1} + \theta_{i+2} - \gamma.$
This gives (9).

To see (7), pick integer $i$ $(0 \le i \le D-3)$
Then by (9),

$$0 = (\theta_i - \beta\theta_{i+1} + \theta_{i+2} - \gamma) - (\theta_{i+1} - \beta\theta_{i+2} + \theta_{i+3} - \gamma)$$

$$= \theta_i - (\beta+1)\theta_{i+1} + (\beta+1)\theta_{i+2} - \theta_{i+3}$$

We have $\beta = \dfrac{\theta_i - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}} - 1 = \dfrac{\theta_i - \theta_{i+1} + \theta_{i+2} - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}}$

as desired.

This proves Claim 7.

We have now proved (3), (4), (7), (8), (9), (11).
Interchanging the roles of $A, A^*$, we obtain (5), (6), (10), (12).

Lecture 30 Mon. April 12, 1993

Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$ with standard module $V$.
Assume $\Gamma$ is $Q$-polynomial w.r.t. the ordering

$$E_0, E_1, \cdots, E_D$$

of primitive idempotents. Let $A_i$ be an $i$-th adjacency matrix and $A = A_1$.

$$A = \sum_{i=0}^{D} \theta_i A_i, \qquad E_1 = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i.$$

Fix $x \in X$, write

$$E_i^* \equiv E_i^*(x), \quad A_i^* \equiv A_i^*(x), \quad A^* \equiv A_1, \quad T \equiv T(x).$$

Then $$A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$$

By Theorem 50. $\exists \beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{R}$ s.t.

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*]$$

$$0 = [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A].$$

Recall <u>raising matrix</u>

$$R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*$$

satisfies

$$R(E_i^* V) \subseteq E_{i+1}^* V \quad (0 \leq i \leq D), \qquad E_{D+1}^* V = 0.$$

<u>lowering matrix</u>

$$L = \sum_{i=0}^{D} E_{i-1}^* A E_i^* \qquad \text{satisfies}$$

$$L E_i^* V \subseteq E_{i-1}^* V \quad (0 \leq i \leq D) \qquad E_{-1}^* V = 0.$$

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flat matrix

$$F = \sum_{i=0}^{D} E_i^* A E_i^*$$

satisfies

$$F E_i^* V \subseteq E_i^* V \qquad (0 \le i \le D)$$

Also

$$A = F + L + R.$$