Theorem 51 With the above notation and assumptions,

(i) $(g_i^- FL^2 + LFL + g_i^+ L^2F - \gamma L^2) E_i^* = 0$ $\quad (2 \le i \le D)$

where

$$g_i^+ = \frac{\theta_{i-2}^* - (\beta+1)\theta_{i-1}^* + \beta\theta_i^*}{\theta_{i-2}^* - \theta_i^*}$$

$$g_i^- = \frac{-\beta\theta_{i-2}^* + (\beta+1)\theta_{i-1}^* - \theta_i^*}{\theta_{i-2}^* - \theta_i^*} \qquad (2 \le i \le D)$$

(ii) $[F, LR - h_i RL] E_i^* = 0$ $\quad (0 \le i \le D)$

where

$$h_i = \frac{\theta_{i-1}^* - \theta_i^*}{\theta_i^* - \theta_{i+1}^*} \qquad (1 \le i \le D-1)$$

and h_0, h_D are indeterminants.

(iii) $(e_i^- RL^2 + (\beta+2) LRL + e_i^+ L^2R$

$+ LF^2 - \beta FLF + F^2L$

$- \gamma(LF+FL) - \delta L) E_i^* = 0$ $\quad (1 \le i \le D)$

where

$$e_i^+ = \frac{\theta_{i-1}^* - (\beta+2)\theta_i^* + (\beta+1)\theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \qquad (1 \le i \le D)$$

$$e_i^- = \frac{-(\beta+1)\theta_{i-2}^* + (\beta+2)\theta_{i-1}^* - \theta_i^*}{\theta_{i-1}^* - \theta_i^*} \qquad (2 \le i \le D)$$

and e_0^+, e_1^- are indeterminants.

Proof

We have

$$0 = A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)$$

(i) Fix i $(2 \le i \le D)$ and multiply the above on the left by E^*_{i-2}, and on the right by E^*_i. Now reduce.

For example

$$E^*_{i-2} A^3 A^* E^*_i = \theta^*_i E^*_{i-2} A^3 E^*_i, \quad \text{where}$$

$$E^*_{i-2} A^3 E^*_i = E^*_{i-2} A \left(\sum_{r=0}^{D} E^*_r\right) A \left(\sum_{s=0}^{D} E^*_s\right) A E^*_i$$

$$= \sum_{r,s} E^*_{i-2} A E^*_r A E^*_s A E^*_i$$

$$= \sum_{\substack{r,s \\ |i-2-r| \le 1,\ |r-s| \le 1\ |s-i| \le 1}} E^*_{i-2} A E^*_r A E^*_s A E^*_i$$

$$= E^*_{i-2} A E^*_{i-2} A E^*_{i-1} A E^*_i$$
$$+ E^*_{i-2} A E^*_{i-1} A E^*_{i-1} A E^*_i$$
$$+ E^*_{i-2} A E^*_{i-1} A E^*_i A E^*_i$$
$$= (FL^2 + LFL + L^2F) E^*_i$$

Reducing the other terms in a similar manner, and simplifying, we obtain (i).

HS

$$E^*_{i-2} A^* A^3 E^*_i = \theta^*_{i-2} E^*_{i-2} A^3 E^*_i$$
$$= \theta^*_{i-2} (FL^2 + LFL + L^2F) E^*_i$$

$$E^*_{i-2} A^2 A^* A E^*_i = (\theta^*_{i-1}(FL^2 + LFL) + \theta^*_i L^2 F) E^*_i$$

$$E^*_{i-2} A A^* A^2 E^*_i = (\theta^*_{i-2} FL^2 + \theta^*_{i-1}(LFL + L^2F)) E^*_i$$

$$E^*_{i-2} (A^2 A^* - A^* A^2) E^*_i = (\theta^*_i - \theta^*_{i-2}) L^2 E^*_i$$

$$E^*_{i-2} (A A^* - A^* A) E^*_i = 0$$

Then we have

$$0=\Big(\big(\theta_i^*-\theta_{i-2}^*\big)(FL^2+LFL+L^2F)$$

$$-(\beta+1)\big(\theta_{i-1}^*(FL^2+LFL)+\theta_i^*L^2F-\theta_{i-2}^*FL^2-\theta_{i-1}^*(LFL+L^2F)\big)$$

$$-\gamma(\theta_i^*-\theta_{i-2}^*)L^2\Big)E_i^*$$

$$=\Big(\big(\theta_i^*-\theta_{i-2}^*-(\beta+1)(\theta_{i-1}^*-\theta_{i-2}^*)\big)FL^2+(\theta_i^*-\theta_{i-2}^*)LFL$$

$$+\big(\theta_i^*-\theta_{i-2}^*-(\beta+1)(\theta_i^*-\theta_{i-1}^*)\big)L^2F-\gamma(\theta_i^*-\theta_{i-2}^*)L^2\Big)E_i^*$$

$$=-(\theta_{i-2}^*-\theta_i^*)\Bigg(\left(\frac{-\beta\theta_{i-2}^*+(\beta+1)\theta_{i-1}^*-\theta_i^*}{\theta_{i-2}^*-\theta_i^*}\right)FL^2+LFL$$

$$+\left(\frac{\theta_{i-2}^*-(\beta+1)\theta_{i-1}^*+\beta\theta_i^*}{\theta_{i-2}^*-\theta_i^*}\right)L^2F-\gamma L^2\Bigg)E_i^*$$

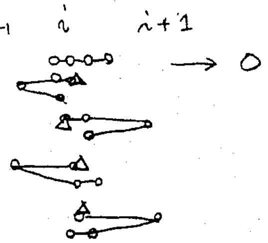
$$=(\theta_i^*-\theta_{i-2}^*)\big(g_i^-FL^2+LFL+g_i^+L^2F-\gamma L^2\big)E_i^*$$

(ii), (iii) are obtained in a similar manner replacing $i-2$ by i (resp. $i-1$).

$\boxed{\text{HS}}$ (ii) $$0=E_i^*\Big(\cancelto{0}{A^3A^*-A^*A^3}-(\beta+1)(A^2A^*A-AA^*A^2)-\gamma\cancelto{0}{(A^2A^*-A^*A^2)}-\delta\cancelto{0}{(AA^*-A^*A)}\Big)E_i^*$$

Since $\beta+1\neq 0$ (by (20) Lec29-8, if $D\geq 3$),

$$0=E_i^*(A^2A^*A-AA^*A^2)E_i^*$$



$$=\Big((\theta_i^*-\theta_{i-1}^*)RLF+(\theta_i^*-\theta_{i+1}^*)LRF$$

$$+(\theta_{i-1}^*-\theta_i^*)FRL+(\theta_{i+1}^*-\theta_i^*)FLR\Big)E_i^*$$

$$=\big[F,\ (\theta_{i-1}^*-\theta_i^*)RL-(\theta_i^*-\theta_{i+1}^*)LR\big]E_i^*$$

$$=(\theta_{i+1}^*-\theta_i^*)\left[F,\ LR-\frac{\theta_{i-1}^*-\theta_i^*}{\theta_i^*-\theta_{i+1}^*}RL\right]E_i^*$$

$$=(\theta_{i+1}^*-\theta_i^*)\big[F,\ LR-h_iRL\big]E_i^*$$

(iii) $0 = E^*_{i-1}\left(A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A)\right] E^*_i$

$$= \Big((\theta^*_i - \theta^*_{i-1})(RL^2 + LRL + L^2R + LF^2 + FLF + F^2L)$$
$$-(\beta+1)\big((\theta^*_{i-1}-\theta^*_{i-2})RL^2 + (\theta^*_{i-1}-\theta^*_i)LRL + (\theta^*_{i+1}-\theta^*_i)L^2R$$
$$+ (\theta^*_i - \theta^*_{i-1})FLF\big)$$
$$-\gamma(\theta^*_i-\theta^*_{i-1})(LF+FL)$$
$$-\delta(\theta^*_i-\theta^*_{i-1})L\Big)E^*_i$$

$RL^2 : i \to i-1 \to i-2 \to i-1$

$LRL : i \to i-1 \to i \to i-1$

$L^2R : i \to i+1 \to i \to i-1$

$LF^2 : i \to i \to i \to i-1$

$FLF : i \to i \to i-1 \to i-1$

$F^2L : i \to i-1 \to i-1 \to i-1$

$LF : i \to i \to i-1$

$FL : i \to i-1 \to i-1$

$$= \Big((\theta^*_i - \theta^*_{i-1} - (\beta+1)(\theta^*_{i-1}-\theta^*_{i-2}))RL^2$$

$(\beta+1)\theta^*_{i-2} - (\beta+2)\theta^*_{i-1} + \theta^*_i$

$$(\theta^*_i - \theta^*_{i-1} - (\beta+1)(\theta^*_{i-1}-\theta^*_i))LRL$$

$-(\beta+2)(\theta^*_{i-1}-\theta^*_i)$

$$(\theta^*_i-\theta^*_{i-1} - (\beta+1)(\theta^*_{i+1}-\theta^*_i))L^2R$$

$-\theta^*_{i-1} + (\beta+2)\theta^*_i - (\beta+1)\theta^*_{i+1}$

$$(\theta^*_i-\theta^*_{i-1})LF^2 + (\theta^*_i-\theta^*_{i-1})F^2L$$

$$(\theta^*_i-\theta^*_{i-1} - (\beta+1)(\theta^*_i-\theta^*_{i-1}))FLF$$

$(\ \beta(\theta^*_{i-1}-\theta^*_i)$

$$-\gamma(\theta^*_i-\theta^*_{i-1})(LF+FL)$$
$$-\delta(\theta^*_i-\theta^*_{i-1})L\Big)E^*_i$$

$$= (\theta^*_i-\theta^*_{i-1})\Big(\frac{-(\beta+1)\theta^*_{i-2}+(\beta+2)\theta^*_{i-1}-\theta^*_i}{\theta^*_{i-1}-\theta^*_i}RL^2 + (\beta+2)LRL$$
$$\frac{\theta^*_{i-1}-(\beta+2)\theta^*_i+(\beta+1)\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i}L^2R + LF^2 - \beta FLF + F^2L$$
$$-\gamma(LF+FL) - \delta L\Big)E^*_i$$

$$= \big(e^-_i RL^2 + (\beta+2)LRL + e^+_i L^2R + LF^2 - \beta FLF + F^2L$$
$$-\gamma(LF+FL) - \delta L\big)E^*_i = 0$$

LEMMA 52 With the notation of Theorem 51.

$$e_i^+ = \frac{\theta_i^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i-1}^*} \qquad (1 \le i \le D-2)$$

$$e_i^- = \frac{\theta_{i-1}^* - \theta_{i-3}^*}{\theta_{i-1}^* - \theta_i^*} \qquad (3 \le i \le D)$$

$$g_i^+ = \frac{\theta_i^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i-2}^*} \qquad (2 \le i \le D-1)$$

$$g_i^- = \frac{\theta_{i-2}^* - \theta_{i-3}^*}{\theta_{i-2}^* - \theta_i^*} \qquad (3 \le i \le D)$$

In particular, $e_i^\pm$, $g_i^\pm$ are non-zero for the range of i given above.

Proof. In each case, equate the above expression with the corresponding expression in Theorem 51. The resulting equation is equal to (7).

$\boxed{\text{HS}}$

$$e_i^+ = \frac{\theta_{i-1}^* - (\beta+2)\theta_i^* + (\beta+1)\theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \qquad \beta = \frac{\theta_{i-1}^* - \theta_i^* + \theta_{i+1}^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i+1}^*}$$

(Cor 45 Lec 26-2, Thm 50 Lec 29-2)

$$e_i^+ = \frac{1}{\theta_{i-1}^* - \theta_i^*}\left(\theta_{i-1}^* - \theta_i^* - (\beta+1)(\theta_i^* - \theta_{i+1}^*)\right) \qquad \beta+1 = \frac{\theta_{i-1}^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i+1}^*}$$

$$= \frac{1}{\theta_{i-1}^* - \theta_i^*}\left(\cancel{\theta_{i-1}^*} - \theta_i^* - \cancel{\theta_{i-1}^*} + \theta_{i+2}^*\right) = \frac{\theta_i^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i-1}^*}$$

$$e_i^- = \frac{1}{\theta_{i-1}^* - \theta_i^*}\left(-(\beta+1)\theta_{i-2}^* + (\beta+2)\theta_{i-1}^* - \theta_i^*\right) \qquad \beta+1 = \frac{\theta_{i-3}^* - \theta_i^*}{\theta_{i-2}^* - \theta_{i-1}^*}$$

$$= \frac{1}{\theta_{i-1}^* - \theta_i^*}\left(\theta_{i-1}^* - \cancel{\theta_i^*} - \theta_{i-3}^* + \cancel{\theta_i^*}\right) = \frac{\theta_{i-1}^* - \theta_{i-3}^*}{\theta_{i-1}^* - \theta_i^*}$$

$$g_i^+ = \frac{1}{\theta_{i-2}^* - \theta_i^*}\left(\theta_{i-2}^* - (\beta+1)\theta_{i-1}^* + \beta\,\theta_i^*\right) \qquad \beta+1 = \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

$$= \frac{1}{\theta_i^* - \theta_{i-2}^*}\left(\theta_i^* - \cancel{\theta_{i-2}^*} + \cancel{\theta_{i-2}^*} - \theta_{i+1}^*\right) = \frac{\theta_i^* - \theta_{i+1}^*}{\theta_i^* - \theta_{i-2}^*}$$

$$g_i^- = \frac{1}{\theta^*_{i-2}-\theta^*_i}\left(-\beta\theta^*_{i-2}+(\beta+1)\theta^*_{i-1}-\theta^*_i\right)$$

$$= \frac{1}{\theta^*_{i-2}-\theta^*_i}\left(\theta^*_{i-2}-\cancel{\theta^*_i}+\cancel{\theta^*_i}-\theta^*_{i-3}\right) = \frac{\theta^*_{i-2}-\theta^*_{i-3}}{\theta^*_{i-2}-\theta^*_i}$$

COROLLARY 53. Let $\Gamma=(X,E)$ be distance regular of diameter $D\geq 3$, Q-polynomial w.r.t $E_0, E_1, \ldots, E_D$. Fix $x\in X$, write $E_i^* \equiv E_i^*(x)$, $R\equiv R(x)$, $L\equiv L(x)$, $F\equiv F(x)$.

Then the following hold.

(i) $FR^2E_j^* \in \mathrm{Span}(RFRE_j^*, R^2FE_j^*, R^2E_j^*)$ $(0\leq j\leq D-3)$

(ii) $R^2FE_j^* \in \mathrm{Span}(RFRE_j^*, FR^2E_j^*, R^2E_j^*)$ $(1\leq j\leq D-2)$

(iii) $LR^2E_j^* \in \mathrm{Span}(RLRE_j^*, R^2LE_j^*, F^2RE_j^*, FRFE_j^*, RF^2E_j^*, RFE_j^*, FRE_j^*, RE_j^*)$ $(0\leq j\leq D-3)$

(iv) $R^2LE_j^* \in \mathrm{Span}(RLRE_j^*, LR^2E_j^*, F^2RE_j^*, FRFE_j^*, RF^2E_j^*, RFE_j^*, FRE_j^*, RE_j^*)$ $(1\leq j\leq D)$

Proof.

Immediate from Theorem 51, Lemma 52. (?)

$\boxed{\text{HS}}$ By Theorem 51, Lemma 52, we have the following but similaly we can obtai above.

(i) $FL^2E_j^* \in \mathrm{Span}(LFLE_j^*, L^2FE_j^*, L^2E_j^*)$ $(3\leq j\leq D)$ $g_j^-\neq 0$

(ii) $L^2FE_j^* \in \mathrm{Span}(LFLE_j^*, FL^2E_j^*, L^2E_j^*)$ $(2\leq j\leq D-1)$ $g_j^+\neq 0$

(iii) $RL^2E_j^* \in \mathrm{Span}(LRLE_j^*, L^2RE_j^*, F^2LE_j^*, FLFE_j^*, LF^2E_j, LFE_j^*, FLE_j^*, LE_j^*)$

↓ $(1\leq j\leq D-2)$ $e_j^+\neq 0$. $(3\leq j\leq D)$ $e_j^-\neq 0$

(iv) $L^2RE_j^* \in \mathrm{Span}(LRLE_j^*, RL^2E_j^*, F^2LE_j^*, FLFE_j^*, LF^2E_j, LFE_j^*, FLE_j^*, LE_j^*)$

Lecture 31 Wed. April 14, 1993.

Let $\Gamma = (X, E)$ be any graph of diameter $D \geq 2$.
Fix $x \in X$. $E_i^* \equiv E_i^*(x)$. $T \equiv T(x)$

Recall adjacency matrix.

$$A = R + L + F$$

$$R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*$$

$$L = \sum_{i=0}^{D} E_{i-1}^* A E_i^*$$

$$F = \sum_{i=0}^{D} E_i^* A E_i^*$$

Observe R is not invertible (indeed $RE_D^* = 0$)
So R^{-1} does not exist.
Below we find a matrix "R^{-1}" $\in T(x)$
s.t. $R^{-1}Rv = v$ for "almost all" $v \in V$.

LEMMA 54 Let $\Gamma = (X, E)$ denote any graph, standard module V over $\mathbb{C}$.
Fix $x \in X$, write
$R \equiv R(x)$, $L \equiv L(x)$, $E_i^* \equiv E_i^*(x)$ $\forall i$.
Then
(i) $\exists$ unique "R^{-1}" $\in \mathrm{Mat}_X(\mathbb{C})$ s.t.
(ia) $R^{-1}v = 0$ if $Lv = 0$ $(v \in V)$
(ib) $R^{-1}RLv = Lv$ $(v \in V)$
(ii) $R^{-1}(E_i^* V) \subseteq E_{i-1}^* V$ $(0 \leq i \leq D)$ $(E_{-1}^* V = 0)$
(iii) $R^{-1} \in \mathrm{Mat}_X(\mathbb{Q})$
(iv) $R^{-1} \in T(x)$

Proof

(i) Consider the orthogonal direct sum.

$$V = (\mathrm{Ker}\, L) + (\mathrm{Ker}\, L)^{\perp}$$

Claim 1 $\quad RL(\mathrm{Ker}\, L)^{\perp} \subseteq (\mathrm{Ker}\, L)^{\perp}$

Proof of Claim 1 $\quad$ Pick $v \in (\mathrm{Ker}\, L)^{\perp}$, and $w \in \mathrm{Ker}\, L$. Show

$$\langle RLv, w \rangle \overset{?}{=} 0$$

But $\quad \overline{R}^{t} = R^{t} = \left(\sum_{i=0}^{D} E_{i+1}^{*} A E_{i}^{*} \right)^{t} = \sum_{i=0}^{D} E_{i}^{*} A E_{i+1}^{*} = L,$

so

$$\begin{aligned} \langle RLv, w \rangle &= \langle Lv, \overline{R}^{t} w \rangle \\ &= \langle Lv, Lw \rangle \\ &= 0. \end{aligned}$$

Claim 2 $\quad RL : (\mathrm{Ker}\, L)^{\perp} \to (\mathrm{Ker}\, L)^{\perp}$

is an isomorphism of vector spaces.

Proof of Claim 2 $\quad$ Suffices to show above map is 1-1.

Suppose $\exists\, v \in (\mathrm{Ker}\, L)^{\perp}$ s.t. $RLv = 0$.

Then

$$\begin{aligned} 0 &= \langle RLv, v \rangle \\ &= \langle Lv, \overline{R}^{t} v \rangle \\ &= \| Lv \|^{2} \end{aligned}$$

So $\quad Lv = 0$.

Hence $\quad v \in \mathrm{Ker}\, L \cap (\mathrm{Ker}\, L)^{\perp} = 0$.

This proves Claim 2.

Now "R^{-1}" denote the unique matrix in $\mathrm{Mat}_X(\mathbb{C})$ such that

$$R^{-1}v = \begin{cases} 0 & \text{if } v \in \operatorname{Ker} L \quad (1) \\ L(RL)^{-1}v & \text{if } v \in (\operatorname{Ker} L)^{\perp} \quad (2) \end{cases}$$

(Observe $(RL)^{-1} : (\operatorname{Ker} L)^{\perp} \longrightarrow (\operatorname{Ker} L)^{\perp}$ exists by Claim 2)

Observe R^{-1} satisfies (ia) (by (1)).

Claim 3 R^{-1} satisfies (ib)

Proof of Claim 3 It suffices to check

$$R^{-1}RLv = Lv$$

for $v \in \operatorname{Ker} L$ and $v \in (\operatorname{Ker} L)^{\perp}$.

The case $v \in \operatorname{Ker} L$ is clear so assume $v \in (\operatorname{Ker} L)^{\perp}$.

Then $RLv \in (\operatorname{Ker} L)^{\perp}$ by Claim 1.

So

$$\begin{aligned} R^{-1}(RLv) &= L(RL)^{-1}RLv \\ &= Lv \end{aligned}$$

as desired.

Uniqueness: Suppose a matrix $\hat{R}^{-1} \in \mathrm{Mat}_X(\mathbb{C})$ satisfied (ia), (ib). Then $\hat{R}^{-1}$ satisfies (1), (2) above.

$\left[\begin{array}{l} (\because)\ (1) \text{ is clear. } (2) \text{ Let } v \in (\operatorname{Ker} L)^{\perp}. \text{ By Claim 2} \\ \exists\, w \in (\operatorname{Ker} L)^{\perp} \text{ s.t } v = RLw. \text{ So } \hat{R}^{-1}v = \hat{R}^{-1}RLw = Lw = L(RL)^{-1}v \end{array} \right]$

Therefore $\hat{R}^{-1}$ agrees with R^{-1} on a basis for V and $\hat{R}^{-1} = R^{-1}$.

Proof of (ii) Pick $v \in E_i^* V$. Show $R^{-1} v \in E_{i-1}^* V$.

WLOG we may assume that

$v \in \operatorname{Ker} L$ or $v \in (\operatorname{Ker} L)^{\perp}$.

If $v \in \operatorname{Ker} L$, then $R^{-1} v = 0 \in E_{i-1}^* V$.

If $v \in (\operatorname{Ker} L)^{\perp}$, then

$$R^{-1} v = L (RL)^{-1} v \in L E_i^* V \subseteq E_{i-1}^* V.$$

Proof of (iii) Observe $R, L \in \operatorname{Mat}_X(\mathbb{Q})$

So V, $\operatorname{Ker} L$, each has basis consisting of vectors in $\mathbb{Q}^{|X|}$.

Repeating the construction of R^{-1} with the base field replaced by $\mathbb{Q}$, we find a matrix $\tilde{R}^{-1} \in \operatorname{Mat}_X(\mathbb{Q})$ satisfying (ia), (ib).

Now R^{-1} and $\tilde{R}^{-1}$ agree on a basis and hence $R^{-1} = \tilde{R}^{-1}$.

Proof of (iv) $RL = L^t L$ is a real symmetric matrix. So it is diagonalizable.

Let θ be any eigenvalue of RL.

Let V_θ denote the corresponding maximal eigenspace in V.

Then
$$V = \sum_{\theta = \text{eigenvalue for } RL} V_\theta \qquad \text{(orthogonal direct sum)}$$

Let $E_\theta : V \to V_\theta$ denote the orthogonal projection.

Then E_θ is a complex polynomial in RL.

Thus $E_\theta \in T(x)$.

$\boxed{\text{HS}}$ E_θ is real. Since RL is an integral matrix, every eigenvalue of RL is an algebraic integer.

Claim 4 $\displaystyle R^{-1} = \sum_{\theta = \text{eigenvalue of } RL,\ \theta \neq 0} \theta^{-1} L E_\theta$ — (*)

In particular, $R^{-1} \in T(x)$.

Proof of Claim 4

Show 2 sides of (*) agree, when applied to arbitrary $v \in V$.

WLOG, $v \in V_\theta$ for some eigenvalue θ of RL.

Let θ' denote any eigenvalue of RL.

$$E_{\theta'} v = \begin{cases} 0 & \text{if } \theta' \neq \theta \\ v & \text{if } \theta' = \theta \end{cases}$$

RHS of (*) applied to v eaquals

$$\begin{cases} 0 & \text{if } \theta = 0 \\ \theta^{-1} L v & \text{if } \theta \neq 0. \end{cases}$$

Show this equals $R^{-1} v$.

Case $\theta = 0$:

Since $RLv = 0$,

$0 = \langle v, RLv \rangle = \| Lv \|^2$

Hence $Lv = 0$ or $v \in \operatorname{Ker} L$.

By (ia), $R^{-1} v = 0$.

Case $\theta \neq 0$:

Since $RLv = \theta v$, $v = \theta^{-1} RLv$.

Hence $R^{-1} v = \theta^{-1} R^{-1} RLv = \theta^{-1} Lv$

by (ib).

No.

Date

Lecture 32, Mon. April 19, 1993

LEMMA 55 Let $\Gamma = (X, E)$ be any graph.
With the notation of LEMMA 54, the following hold.

(i) Let W denote a thin irreducible T-module with endpoint r, diameter d.
Pick i $(0 \leq i \leq d)$, pick $v \in E^*_{r+i} W$.
Then

$$R^{-1} R v = \begin{cases} v & \text{if } i < d \\ 0 & \text{if } i = d. \end{cases}$$

(ii) Assume Γ is distance regular and thin w.r.t. x. Pick t $(0 \leq t < D/2)$, and pick $v \in E^*_t V$. Then

$$R^{-1} R^i v = R^{i-1} v \qquad (1 \leq i \leq D-2t).$$

In particular, $R^{-1} R v = v$.

(iii) Assume Γ is distance regular and thin w.r.t. x.
Then

$$R : E^*_i V \to E^*_{i+1} V$$

is one-to-one $\quad (0 \leq i < D/2)$

Proof.

(i) Let $w_0, w_1, \cdots, w_d$ be a basis for W

$w_i \in E^*_{r+i} W$

$$R w_i = w_{i+1} \qquad (0 \leq i < d)$$

$$L w_i = x_i(W) w_{i-1} \qquad (1 \leq i \leq d)$$

So

$$R L w_i = x_i(W) w_i \qquad (1 \leq i \leq d)$$

(See Lecture 9, Lemma 15)

$$R^{-1} R w_i = ?$$

If $i = d$, $R^{-1} R w_d = 0$.

If $0 \leq i < d$,

$$R^{-1}Rw_i = R^{-1}w_{i+1}$$
$$= x_{i+1}(W)^{-1}R^{-1}RLw_{i+1}$$
$$= x_{i+1}(W)^{-1}Lw_{i+1}$$
$$= x_{i+1}(W)^{-1}x_{i+1}(W)\,w_i$$
$$= w_i.$$

Thus we have (i).

$\boxed{\text{HS}}$

$$RLw_i = Rx_i(W)w_{i-1} = x_i(W)w_i$$
$$LRw_i = Lw_{i+1} = x_{i+1}(W)w_i$$
$$[L,R]w_i = (x_{i+1}(W) - x_i(W))w_i \qquad 0 \leq i \leq d.$$
$$x_0(W) = 0, \quad x_{d+1}(W) = 0$$
$$[L,R]|_W = \sum_{i=0}^{d}(x_{i+1}(W) - x_i(W))E^*_{r+i}|_W$$

(ii) $V = \sum W$ (orthogonal direct sum of thin irreducible T-modules)

Then $E^*_t V = \sum_{r(W)\leq t} E^*_t W$ (orthogonal direct sum)

WLOG we may assume that
$v \in E^*_t W$

for some thin irreducible T-module with endpoint at most t.

Now if $i \leq D - 2t$, then

$$t + i \leq D - t$$
$$\leq D - r(W)$$
$$\leq r(W) + d(W) \qquad \text{by Lemma 28 (iii)}$$
$$(D \leq 2r + d)$$

So $\quad t+i-1 \leq r(W)+d(W)-1$

Hence

$$R^{-1}R^{i}v = R^{-1}R(R^{i-1}v) \qquad (R^{i-1}v \in E^*_{t+i-1}W)$$
$$= R^{i-1}v \qquad \text{by (i)}$$

(iii) Suppose $Rv=0$ for some $v \in E^*_i V$ $(0 \leq i < D/2)$.

$$0 = R^{-1}Rv = v \qquad (\text{by (ii) with } t=i \text{ ad } i=1)$$

DEF. Let $\Gamma = (X, E)$ denote any graph with the standard module V.
Fix $x \in X$. Write $E^*_i \equiv E^*_i(x)$, $T \equiv T(x)$, $L \equiv L(x)$

DEF 1. For $\forall i$ $(0 \leq i \leq D)$, define subspace
$V_i := V_i(x) \subseteq V$ by

$$V_i = \sum W,$$

where the sum being over irreducible T-modules W with end point i.

Observe :

$$V = V_0 + V_1 + \cdots + V_D \qquad \text{(orthogonal direct sum)}$$
$$V_0 = \text{trivial } T\text{-module}$$

DEF 2. $(E^*_i V)_{new} \equiv E^*_i V_i \qquad (0 \leq i \leq D)$

In general $\quad (E^*_i V)_{new} \subseteq \operatorname{Ker} L \cap E^*_i V \subseteq \operatorname{Ker}(L \cdot E^*_i)$.

If each irreducible T-module with endpoint $< i$ is thin,

$$(E_i^* V)_{new} = \operatorname{Ker} L \cap E_i^* V \subseteq \operatorname{Ker}(L \cdot E_i^*).$$

$\boxed{\text{HS}}$ $E_i^* V = \sum_{j<i} V_j + V_i$

For V_j part, take $w_{i-j} \in W$: irreducible with endpoint $j < i$. Then

$$L w_{i-j} = x_{i-j}(W)\, w_{i-j-1} \neq 0$$

and $L|_{\sum_{j<i} E_i^* V_j} : \sum_{j<i} E_i^* V_j \to V$

is one to one.

(originally 54)

LEMMA 56 Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$. Fix $x \in X$.
$R \equiv R(x)$, $L \equiv L(x)$, $F \equiv F(x)$.

Pick $v \in (E_1^* V)_{new}$.

Then

(i) $R E_i^* A_{i-1} v = c_i E_{i+1}^* A_i v$ $(1 \leq i \leq D)$

(ii) $F E_i^* A_{i-1} v = R E_{i-1}^* A_i v + (a_{i-1} - c_i + c_{i-1}) E_i^* A_{i-1} v - c_i E_i^* A_{i+1} v$ $(1 \leq i \leq D)$

(iii) $L E_i^* A_{i-1} v = F E_{i-1}^* A_i v - (a_{i-1} - c_i + c_{i-1}) E_{i-1}^* A_i v + b_{i-1} E_{i-1}^* A_{i-2} v$ $(2 \leq i \leq D)$

(iv) $L E_i^* A_{i+1} v = b_i E_{i-1}^* A_i v$ $(1 \leq i \leq D-1)$

Proof.

(i) Let $v = \sum_{y \in X\ \partial(x,y)=1} \alpha_y \hat{y}$ for some $\{\alpha_y\} \subseteq \mathbb{C}$

Then

$$Lv = \Big(\sum_{y \in X\ \partial(x,y)=1} \alpha_y\Big)\hat{x} = 0$$

So

$$\sum_{y \in X\ \partial(x,y)=1} \alpha_y = 0$$

Thus.

$$v = \sum_{y \in X,\ \partial(x,y)=1} \alpha_y(\hat{y} - \hat{x}).$$

Let $\tilde{A}_i = A_0 + A_1 + \cdots + A_i \qquad (0 \le i \le D)$

Then

$$\tilde{A}_i v = \sum_{\substack{y \in X \\ \partial(x,y)=1}} \alpha_y \tilde{A}_i(\hat{y} - \hat{x})$$

$$= \sum_{y \in X\ \partial(x,y)=1} \alpha_y \Big(\sum_{z \in X\ \partial(y,z)=i\ \partial(x,z)=i+1} \hat{z} - \sum_{z' \in X\ \partial(y,z')=i+1,\ \partial(x,z')=i} \hat{z}' \Big)$$

[$\boxed{\text{HS}}$ See Lec 16-5. $\therefore\ \tilde{A}_D v = 0$]

$$= \sum_{y \in X,\ \partial(x,y)=1} \alpha_y \big(E^*_{i+1} A_i \hat{y} - E^*_i A_{i+1} \hat{y}\big)$$

$$= E^*_{i+1} A_i v - E^*_i A_{i+1} v$$

Recall (Lec16-5 : Claim 1 in the proof of Thm 32)

$$A\tilde{A}_i = c_{i+1}\tilde{A}_{i+1} + (a_i - c_{i+1} + c_i)\tilde{A}_i + b_i \tilde{A}_{i-1} \qquad (0 \le i \le D-1)$$

(This is valid for $i=0$ as $A\tilde{A}_0 = AI \stackrel{?}{=} c_1\tilde{A} - \tilde{A}_0 = A$.)
by setting $\tilde{A}_{i-1} = 0$.)

Now (i) – (iv) are obtained by applying this to v on the right and multiplied by E_j^* $(0 \le j \le D)$ on the left.

$\boxed{\text{HS}}$ $A\tilde{A}_{i-1} v = AE_i^* A_{i-1} v - AE_{i-1}^* A_i v$ equals (for $1 \le i \le D$)

$$(c_i \tilde{A}_i + (a_{i-1} - c_i + c_{i-1}) \tilde{A}_{i-1} + b_{i-1} \tilde{A}_{i-2}) v$$

$$= c_i E_{i+1}^* A_i v - c_i E_i^* A_{i+1} v$$

$$+ (a_{i-1} - c_i + c_{i-1}) E_i^* A_{i-1} v - (a_{i-1} - c_i + c_{i-1}) E_{i-1}^* A_i v$$

$$+ b_{i-1} E_{i-1}^* A_{i-2} v - b_{i-1} E_{i-2}^* A_{i-1} v$$

(i) $RE_i^* A_{i-1} v = E_{i+1}^* A E_i^* A_{i-1} v = c_i E_{i+1}^* A_i v$ $\quad (1 \le i \le D)$

(ii) $FE_i^* A_{i-1} v = E_i^* A E_i^* A_{i-1} v$

$$= RE_{i-1}^* A_i v - c_i E_i^* A_{i+1} v + (a_{i-1} - c_i + c_{i-1}) E_i^* A_{i-1} v \quad (1 \le i \le D)$$

(iii) $LE_i^* A_{i-1} v = E_{i-1}^* A E_i^* A_{i-1} v$

$$= FE_{i-1}^* A_i v - (a_{i-1} - c_i + c_{i-1}) E_{i-1}^* A_i v + b_{i-1} E_{i-1}^* A_{i-2} v \quad (2 \le i \le D)$$

(Even if $i = 1$, this is valid by setting $A_{i-2} = 0$.)

(iv) $LE_i^* A_{i+1} v = E_{i-1}^* A E_i^* A_{i+1}$

$$= b_i E_{i-1}^* A_i v \quad (1 \le i \le D-1)$$

LEMMA 57 (originally 55)

Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$. Fix $x \in X$, $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, $R = R(x)$, $F = F(x)$, $L = L(x)$.

For $\forall v \in (E_1^* V)_{new}$, the following are equivalent

(i) $E_i^* A_{i-1} v$, $E_i^* A_{i+1} v$ are linearly dependent for every i $(1 \leq i \leq D-1)$

(ii) There exists a thin irreducible T-module W with endpoint 1 that contains v.

If (i) (ii) hold, then

$$W = \mathrm{Span}(E_1^* A_0 v, E_2^* A_1 v, \cdots, E_D^* A_{i-1} v).$$

Proof.

(ii) → (i) Clear as

$$E_i^* A_{i-1} v,\ E_i^* A_{i+1} v \in E_i^* W = \mathrm{Span}(w_{i-1})$$

(i) → (ii) Consider the sequence

$$E_1^* A_0 v,\ E_2^* A_1 v,\ E_3^* A_2 v, \cdots\cdots, E_{D+1}^* A_D v$$

The first term is nonzero and the last term is 0

So there exists

$$n := \min\{ i \mid 1 \leq i \leq D,\ E_{i+1}^* A_i v = 0 \}.$$

Now

$$E_{j+1}^* A_j v = 0 \qquad (n \leq j \leq D) \quad \text{—— (1)}$$

[$\boxed{\text{HS}}$ Use induction and Lemma 56 (i) $\quad E_{j+1}^* A_j v \in \mathrm{Span}(R E_j^* A_{j-1} v)$ $(j \geq 1)$]

By our assumption (i) and the definition of n,

$$E_j^* A_{j+1} v \in \mathrm{Span}(E_j^* A_{j-1} v) \neq 0 \qquad (1 \le j \le n)$$

By Lemma 56 (i)

$$R E_j^* A_{j-1} v \in \mathrm{Span}(E_{j+1}^* A_j v) \qquad (1 \le j \le n)$$

By Lemma 56 (ii)

$$\begin{aligned} F E_j^* A_{j-1} v &\in \mathrm{Span}(\underbrace{R E_{j-1}^* A_j v}_{=0 \text{ if } j=1},\ E_j^* A_{j-1} v,\ E_j^* A_{j+1} v) \\ &\subseteq \mathrm{Span}(R E_{j-1}^* A_{j-2} v,\ E_j^* A_{j-1} v) \\ &\subseteq \mathrm{Span}(E_j^* A_{j-1} v) \qquad (1 \le j \le n) \end{aligned}$$

By Lemma 56 (iii)

$$\begin{aligned} L E_j^* A_{j-1} v &\in \mathrm{Span}(F E_{j-1}^* A_j v,\ E_{j-1}^* A_j v,\ E_{j-1}^* A_{j-2} v) \\ &\subseteq \mathrm{Span}(F E_{j-1}^* A_{j-2} v,\ E_{j-1}^* A_{j-2} v) \\ &\subseteq \mathrm{Span}(E_{j-1}^* A_{j-2} v) \qquad (2 \le j \le n) \end{aligned}$$

Hence

$$W = \mathrm{Span}(E_1^* A_0 v,\ E_2^* A_1 v,\ \cdots,\ E_n^* A_{n-1} v)$$

is R, F, L invariant.

Therefore W is a thin T-module with endpoint 1 that contains v.

Lecture 33 Wed April 21, 1993

LEMMA 58 (originally 56)

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme

Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $M^* \equiv M^*(x)$, $T \equiv T(x)$

Then the following hold

(i) $E_0^* M M^* = E_0^* M$ | (i') $E_0 M^* M = E_0 M^*$

(ii) $E_0^* T = E_0^* M$ | (ii') $E_0 T = E_0 M$

(iii) $T E_0^* T = M E_0^* M$ | (iii') $T E_0 T = M^* E_0 M^*$

(iv) $E_0^* E_0 E_0^* = |X|^{-1} E_0^*$ | (iv') $E_0 E_0^* E_0 = |X|^{-1} E_0$

(v) lines (i)–(iv) hold if we interchange

(E_0, E_0^*), (M, M^*).

Moreover, $M E_0^* M = M^* E_0 M^*$.

Proof

(i) $\supseteq$: $1 \in M^*$ implies $M \subseteq M M^*$

$\subseteq$ Pick $\alpha \in E_0^* M M^*$. Show $\alpha \in E_0^* M$

Since $A_0, A_1, \cdots, A_D$ span M, and since

$E_0^*, \cdots, E_D^*$ span M^*

WLOG

$$\alpha = E_0^* A_i E_j^*$$

for some i, j $(0 \le i, j \le D)$

WLOG

$$i = j$$

else $\alpha = 0$ by Lemma 35 (Lec 20-3)

($E_h^* A_i E_j^* \ne 0 \Leftrightarrow p_{hi}^j \ne 0$)

Now $\alpha = E_0^* A_i \left(\sum_{h=0}^{D} E_h^* \right)$

$$= E_0^* A_i$$

$$\in E_0^* M$$

(ii) $\supseteq$: This is clear

$\subseteq$: $E_0^* T$ is the minimal right ideal of T containing E_0^*.

So we just have to show that $E_0^* M$ is a right ideal of T containing E_0^*

It clearly contains E_0^* since $I \in M$, and is a right ideal of T by (i) and the fact that T is generated by M and M^*.

(iii) By the transpope of (ii),

$$T E_0^* = M E_0^*,$$

so

$$\begin{aligned} T E_0^* T &= (T E_0^*)(E_0^* T) \\ &= M E_0^* E_0^* M \\ &= M E_0^* M \end{aligned}$$

(iv)

$$\begin{aligned} E_0^* E_0 E_0^* &= \frac{1}{|X|} E_0^* \left(\sum_{h=0}^{D} A_h \right) E_0^* \\ &= |X|^{-1} E_0^* A_0 E_0^* \\ &= |X|^{-1} E_0^* \end{aligned}$$

(v). The first part is clear by using

Lemma 35 (ii) $E_h A_i^* E_j \neq 0 \Leftrightarrow q_{hi}^j \neq 0$ (Lec20-3)

Lemma 40 (iii) (Lec22-1) $q_{0i}^j = \delta_{ij}$

Also

$$\begin{aligned} M E_0^* M &= T E_0^* T \\ &= T E_0^* E_0 E_0^* T \\ &\subseteq T E_0 T \\ &= M^* E_0 M^*. \end{aligned}$$

and $\quad M^* E_0 M^* \subseteq M E_0^* M \quad$ by dual argument.

So $\quad M^* E_0 M^* = M E_0^* M$.

This proves the lemma.

(originally 57)

LEMMA 59 $\quad$ Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$, Q-polynomial w.r.t. $E_0, E_1, \cdots, E_D$. Pick $x \in X$

write $\quad E_i^* \equiv E_i^*(x)$, $\quad M^* \equiv M^*(x)$, $\quad T \equiv T(x)$

(i) $\quad E_1^* M M^* = E_1^* M + E_1^* E_0 M^* + E_1^* E_1 M^*$

(ii) $\quad E_1 M^* M = E_1 M^* + E_1 E_0^* M + E_1 E_1^* M$

Proof

(i) $\quad$ View E_{-1}^*, E_{D+1}^* as 0.

View θ_{-1}^*, θ_{D+1}^* as indeterminates.

Let $\triangle$ denote RHS in (i).

$\supseteq$: $\quad I \in M^*$ implies $M \subseteq M M^*$.

$\subseteq$: $\quad$ Suppose not. Then there exists

$$\alpha \in E_1^* M M^* \setminus \triangle \tag{1}$$

Since $A_0, A_1, \cdots, A_D$ span M, since $E_0^*, \cdots, E_D^*$ span M^*, WLOG

$$\alpha = E_1^* A_i E_j^*$$

for some i, j $\quad (0 \leq i, j \leq D)$.

Observe $\quad |i - j| \leq 1$.

else $\quad \alpha = 0 \quad$ by Lemma 35 (Lec 20-3)

WLOG assume $i + j$ is minimal subject to the above constraints.

First assume

$$j = i+1 \qquad (2)$$

Observe

$$\begin{aligned} & E_1^* A_i E_{i+1}^* + E_1^* A_i E_i^* + E_1^* A_i E_{i-1}^* \\ &= E_1^* A_i \left(\sum_{h=0}^{D} E_h^* \right) \\ &= E_1^* A_i \\ &\in \Delta. \end{aligned} \qquad (3)$$

Also observe

$$E_1^* A_i E_i^*, \quad E_1^* A_i E_{i-1}^* \in \Delta$$

by the minimality of $i+j$, so

$$\alpha = E_1^* A_i E_{i+1}^* \in \Delta$$

by (3). Hence (2) cannot occur.

Since $|i-j| \leq 1$,

$$i \in \{j, j+1\} \qquad (4)$$

Observe

$$\begin{aligned} & E_1^* A_{j+1} E_j^* + E_1^* A_j E_j^* + E_1^* A_{j-1} E_j^* \\ &= E_1^* \left(\sum_{h=0}^{D} A_h \right) E_j^* \\ &= |X| \; E_1^* E_0 E_j^* \\ &\in \Delta \end{aligned} \qquad (5)$$

and

$$\begin{aligned} & \theta_{j+1}^* E_1^* A_{j+1} E_j^* + \theta_j^* E_1^* A_j E_j^* + \theta_{j-1}^* E_1^* A_{j-1} E_j^* \\ &= E_1^* \left(\sum_{h=0}^{D} \theta_h^* A_h \right) E_j^* \\ &= |X| \; E_1^* E_1 E_j^* \\ &\in \Delta \end{aligned} \qquad (6)$$

Since $E_1^* A_{j-1} E_j^* \in \Delta$

by the minimality of $i+j$, so

$$E_1^* A_{j+1} E_j^* + E_1^* A_j E_j^* \in \Delta$$

$$\theta_{j+1}^* E_1^* A_{j+1} E_j^* + \theta_j^* E_1^* A_j E_j^* \in \Delta$$

But $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ are distinct by Lemma 41 (iv) (Lec22-4),

so

$$E_1^* A_{j+1} E_j^*, \quad E_1^* A_j E_j^* \in \Delta$$

But α is one of these two matrices,

so $\alpha \in \Delta$

Hence (4) cannot occur either, and we have a contradiction.

(ii) dual argument.

LEMMA 60 (originally Lemma 57.5)

With the above notation, set

$$\tilde{J} := E_1^* J E_1^*$$
$$\tilde{A} := E_1^* A E_1^*$$

(i) $\tilde{J}^2 = k\tilde{J}$ (k = valency of Γ)

(ii) $\tilde{J}\tilde{A} = \tilde{A}\tilde{J} = a_1\tilde{J}$ ($a_1 = p^1_{11}$ for Γ)

(iii) $E_1^* E_0 E_1^* = |X|^{-1}\tilde{J}$

(iv) $E_1^* E_1 E_1^* = |X|^{-1}\left(E_1^*(\theta_0^* - \theta_2^*) + \tilde{A}(\theta_1^* - \theta_2^*) + \tilde{J}(\theta_2^*)\right)$

Proof.

(i) The first subconstituent has k vertices.

(ii) The first subconstituent is regular of valency a_1

(iii) Since $E_0 = |X|^{-1}J$,
$E_1^* E_0 E_1^* = |X|^{-1}\tilde{J}$.

(iv)
$$E_1^* E_1 E_1^* = E_1^*\left(|X|^{-1}\sum_{h=0}^{D} \theta_h^* A_h\right)E_1^*$$
$$= |X|^{-1}\left(\theta_0^* E_1^* A_0 E_1^* + \theta_1^* E_1^* A_1 E_1^* + \theta_2^* E_1^* A_2 E_1^*\right)$$
$$= |X|^{-1}\left(\theta_0^* E_1^* + \theta_1^* \tilde{A} + \theta_2^* E_1^* A_2 E_1^*\right) \quad —(1)$$

Also

$$\tilde{J} = E_1^* J E_1^*$$
$$= E_1^* A_0 E_1^* + E_1^* A_1 E_1^* + E_1^* A_2 E_1^*$$
$$= E_1^* + \tilde{A} + E_1^* A_2 E_1^* \quad —(2)$$

Eliminating the $E_1^* A_2 E_1^*$ term in (1) using equation (2) we get (iv)

LEMMA 61 (originally Lemma 58)

With the above notation,

(i) $E_1^*T = E_1^*E_0M^* + E_1^*M + E_1^*E_1M^* + E_1^*E_1E_1^*M + \cdots$

(ii) $E_1^*TE_1^* = \mathrm{Span}(E_1^*E_0E_1^*,\ E_1^*,\ E_1^*E_1E_1^*,\ (E_1^*E_1E_1^*)^2, \cdots)$

(iii) $E_1^*TE_1^* = \mathrm{Span}(\tilde{J},\ E_1^*,\ \tilde{A},\ \tilde{A}^2, \cdots)$

(iv) $E_1^*TE_1^*$ is symmetric (in particular, commutative)

Proof.

(i) $\supseteq$: clear

$\subseteq$: E_1^*T is the minimal right ideal of T that contains E_1^*.
RHS contains E_1^*, so show RHS is a right ideal of T.
Show RHS is closed w.r.t. multiplication on right by M, M^*.

$E_1^*E_0M^*(M) = E_1^*E_0M^*$ by dual of Lemma 58.(i)
$E_1^*E_0M^*(M^*) = E_1^*E_0M^*$ (Lec 33-1)

$E_1^*E_1E_1^* \cdots E_1^*M(M^*)$

$= E_1^*E_1E_1^* \cdots E_1(E_1^*MM^*)$ ← Spa

$= E_1^*E_1E_1^* \cdots E_1(E_1^*M + E_1^*E_0M^* + E_1^*E_1M^*)$

by LEMMA 59 (Lec 33-3)

$\in$ RHS

(because
$E_1^*E_1E_1^* \cdots E_1E_1^*E_0M^*$
$\subseteq E_1^*TE_0T = E_1^*M^*E_0M^* = E_1^*E_0M^*$.)

$E_1^* E_1 \cdots \quad E_1 M^* (M)$

$= E_1^* \cdots \quad E_1^* (E_1 M^* M)$

$= E_1^* \cdots \quad E_1^* (E_1 M^* + E_1 E_0^* M + E_1 E_1^* M)$

by LEMMA 59 (Lec 33-3)

$\in$ RHS

because

$$E_1^* \cdots E_1^* E_1 E_0^* M \subseteq E_1^* T E_0^* T$$
$$= E_1^* M E_0^* M$$
$$= E_1^* M^* E_0 M^*$$
(by LEMMA 58 (Lec 33-1) the last part)
$$= E_1^* E_0 M^*$$

(ii) Multiply (i) on the right by E_1^*, we have

$$E_1^* T E_1^* = E_1^* E_0 M^* E_1^* + E_1^* M E_1^* + E_1^* E_1 M^* E_1^* + \cdots + E_1^* E_1 \cdots E_1 M^* E_1^* + E_1^* E_1 \cdots E_1^* M E_1^*$$
$$= \mathrm{Span}(E_1^* E_0 E_1^*, \ E_1^*, \ E_1^* E_1 E_1^*, \ (E_1^* E_1 E_1^*)^2, \cdots)$$

[$\boxed{\text{HS}}$ because $E_1^* M E_1^* = \mathrm{Span}(E_1^* A_0 E_1^*, E_1^* A_1 E_1^*, E_1^* A_2 E_1^*)$
by Lemma 50 (Lec 33-6) $= \mathrm{Span}(E_1^*, E_1^* E_1 E_1^*, E_1^* E_0 E_1^*)$. Moreover
$E_1^* \cdots E_1^* E_0 E_1^* \subseteq E_1^* T E_0 T E_1^* = E_1^* M^* E_0 M^* E_1^* \in \mathrm{Span}(E_1^* E_0 E_1^*)$]

(iii) By (ii), $E_1^* T E_1^*$ is generated by $\tilde{J} = |X| E_1^* E_0 E_1^*$ and $E_1^* E_1 E^*$.
By LEMMA 60 (iv) (Lec 33-6).
$E_1^* T E_1^*$ is generated by $\tilde{J}, \hat{A}$.
But $\mathrm{Sp}\,\tilde{J}$ is a 2-sided ideal by
LEMMA 60 (i) (ii) (Lec 33-6).
Hence we have (iii)

(iv) $\hat{A}, \tilde{J}$ are symmetric commuting matrices.
we have the clai.

Lecture 34 Fri. April 23, 1993

Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$.
Assume Γ is Q-polynomial w.r.t. $E_0, E_1, \cdots, E_D$.
Write $\tilde{A}_i = A_0 + A_1 + \cdots + A_i \quad (0 \leq i \leq D)$
Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $M^* \equiv M^*(x)$, $T \equiv T(x)$.

Pick $0 \neq v \in (E_1^* V)_{new}$. Set $v^* = |X| E_1 v$.
Will show
$$Tv = Mv + M^* v^*$$
We need preliminary lemma

LEMMA 62 (originally Lemma 59).
With the above notation,

(i) $\tilde{A}_h v = E_{h+1}^* A_h v - E_h^* A_{h+1} v \quad (0 \leq h \leq D) \ (E_{D+1}^* = A_{D+1} = 0)$

(ii) $E_h^* v^* = (\theta_{h-1}^* - \theta_h^*) E_h^* A_{h-1} v - (\theta_h^* - \theta_{h+1}^*) E_h^* A_{h+1} v$
$$(0 \leq h \leq D) \qquad (A_{-1} = A_{D+1} = 0)$$

(iii) $(\theta_i^* - \theta_{i+1}^*) E_{i+1}^* A_i v$
$$= \left(\sum_{h=0}^{i} (\theta_h^* - \theta_{i+1}^*) A_h \right) v - \left(\sum_{h=0}^{i} E_h^* \right) v^* \qquad (0 \leq i \leq D-1)$$

(iv) $(\theta_i^* - \theta_{i+1}^*) E_i^* A_{i+1} v$
$$= \left(\sum_{h=0}^{i-1} (\theta_h^* - \theta_i^*) A_h \right) v - \left(\sum_{h=0}^{i} E_h^* \right) v^* \qquad (0 \leq i \leq D-1)$$

(v) $Mv + M^* v^* = \mathrm{Span}\{ E_i^* A_{i-1} v, \ E_{i-1}^* A_i v \mid 1 \leq i \leq D \}$

Proof

(i) See Lec 32-5.
It is already done in Lemma 56. (Lec 32-5)

(ii) $$E_h^* v^* = |X| E_h^* E_1 v$$
$$= E_h^* \left(\sum_{i=0}^{D} \theta_i^* A_i \right) v$$
$$= E_h^* \left(\sum_{i=0}^{D} \theta_i^* \left(\tilde{A}_i - \tilde{A}_{i-1} \right) \right) v$$
$$= E_h^* \left(\sum_{i=0}^{D-1} (\theta_i^* - \theta_{i+1}^*) \tilde{A}_i \right) v + E_h^* \theta_D^* \underbrace{\tilde{A}_D v}_{=0}$$
$$= E_h^* \left(\sum_{i=0}^{D-1} (\theta_i^* - \theta_{i+1}^*)(E_{i+1}^* A_i v - E_i^* A_{i+1} v) \right)$$
$$= (\theta_{h-1}^* - \theta_h^*) E_h^* A_{h-1} v - (\theta_h^* - \theta_{h+1}^*) E_h^* A_{h+1} v$$

(iii), (iv) Call equation in (iii) i^+ and call equation in (iv) i^-.
Prove in order, $0^-, 0^+, 1^-, 1^+, 2^-, 2^+, \cdots$

0^- : trivial

$\boxed{\text{HS}}$ LHS $= (\theta_0^* - \theta_1^*) E_0^* A_1 v$ $\quad (\in Lv = 0)$

$= (\theta_{-1}^* - \theta_1^*) E_0^* A_{-1} v - E_h^* v^*$ by (ii)

$= -E_0^* v^*$ $\quad$ (= 0 otherwise Tv has end point 0)

$=$ RHS

i^+ ; using (i) and i^-

LHS $= (\theta_i^* - \theta_{i+1}^*) E_{i+1}^* A_i v$

$= (\theta_i^* - \theta_{i+1}^*) E_i^* A_{i+1} v + (\theta_i^* - \theta_{i+1}^*) \tilde{A}_i v$ $\quad$ (by (i))

$= \left(\sum_{h=0}^{i-1} (\theta_h^* - \theta_i^*) A_h \right) v - \left(\sum_{h=0}^{i} E_h^* \right) v^* + (\theta_i^* - \theta_{i+1}^*) \left(\sum_{h=0}^{i} A_h \right) v$

(by i^-)

$$= \left(\sum_{h=0}^{i} (\theta_h^* - \theta_{i+1}^*) A_h\right) v - \left(\sum_{h=0}^{i} E_h^*\right) v^*$$

i^- : using (ii) and $(i-1)^+$

$(\theta_i^* - \theta_{i+1}^*) E_i A_{i+1} v$

$$= (\theta_{i-1}^* - \theta_i^*) E_i^* A_{i-1} v - E_i^* v^* \qquad \text{(by (ii))}$$

$$= \left(\sum_{h=0}^{i-1} (\theta_h^* - \theta_i^*) A_h\right) v - \left(\sum_{h=0}^{i-1} E_h^*\right) v^* - E_i^* v^*$$

$$= \left(\sum_{h=0}^{i-1} (\theta_h^* - \theta_i^*) A_h\right) v - \left(\sum_{h=0}^{i} E_h^*\right) v^*$$

(v) Immediate from (i) − (iv).

$\boxed{\text{HS}}$ $Mv + M^*v^* \subseteq \operatorname{Span}\{\tilde{A}_h v, E_h^* v^* \mid 0 \le h \le D\}$

$\subseteq \operatorname{Span}\{E_h^* A_{h-1} v, E_{h-1}^* A_h v \mid 1 \le h \le D\}$

by (i) and (ii)

On the other hand,

$E_h^* A_{h-1} v,\ E_{h-1}^* A_h v \in Mv + M^* v^* \quad (1 \le h \le D)$

by (iii) and (iv)

LEMMA 63 (originally 60)

With the notation of Lemma 62, assume $0 \neq v \in (E_1^* V)_{new}$ is an eigenvector for $\tilde{A} := E_1^* A E_1^*$.

Then

(i) $Tv = Mv + M^* v^*$, where $v^* = |X| E_1 v$

(ii) $Tv = \mathrm{Span}\{v_1^+, v_2^+, \cdots, v_D^+, v_2^-, v_3^-, \cdots, v_{D-1}^-\}$,

where $v_i^+ = E_i^* A_{i-1} v$, $v_i^- = E_i^* A_{i+1} v$

(iii) $\dim E_1^* Tv = 1$

$\dim E_i^* Tv \leq 2$ $\quad (2 \leq i \leq D-1)$

$\dim E_D^* Tv \leq 1$

(iv) Tv is an irreducible T-module.

Proof

(i) $\supseteq$: $v \in Tv$. So $Mv \subseteq Tv$ and $v^* \in Mv \subseteq Tv$.

Hence $M^* v^* \subseteq Tv$.

$\subseteq$: It suffices to show that $Mv + M^* v^*$ is a T-module (since it clearly contains v).

Show

(a) $M^* M v \subseteq Mv + M^* v^*$

(b) $M M^* v^* \subseteq Mv + M^* v^*$.

Proof of (a) ;

By the transpose of (i) in Lemma 59 (Lec33-3)

$$M^*ME_1^* = ME_1^* + M^*E_0E_1^* + M^*E_1E_1^*$$

Since $v \in E_1^*V$, $E_1^*v = v$ and

$$M^*Mv = Mv + M^*E_0v + M^*E_1v$$

But also $E_0v = 0$ since v is orthogonal to the trivial T-module.

Since $E_1v = |X|^{-1}v^*$,

$$M^*Mv = Mv + M^*v^*$$

as desired.

(b). is obtained from the transpose of (ii) in Lemma 59 (Lec 33-3)

HS
$$\begin{aligned} MM^*v^* &= MM^*E_1v^* \\ &= M^*E_1v^* + ME_0^*E_1v^* + ME_1^*E_1v^* \\ &= M^*v^* + ME_0^*v^* + ME_1^*v^* \end{aligned}$$

$E_0^*v^* \in Tv$ and $E_0^*Tv = 0$ as $v \in (E_1^*V)_{new}$

So $E_0^*v^* = 0$.

$$\begin{aligned} E_1^*v^* &= |X|E_1^*E_1v = |X|E_1^*E_1E_1^*v \\ &= ((\theta_0^* - \theta_2^*)E_1^* + (\theta_1^* - \theta_2^*)E_1^*AE_1^* + \theta_2|X|E_1^*E_0E_1^*)v \\ &= (\theta_0^* - \theta_2^*)v + (\theta_1^* - \theta_2^*)E_1^*AE_1^*v + \theta_2|X|E_1^*E_0v \\ &\in \mathrm{Span}\{v\} \end{aligned}$$

as $E_0v = 0$ and v is an eigenvector of $E_1^*AE_1^*$

☆ $v \in (E_1^*V)_{new}$. If v is an eigenvector of $E_1^*AE_1^*$, $E_1^*v^* \in \mathrm{Span}\{v\}$

(ii) $$\begin{aligned} T v &= M v + M^* v^* \\ &= \mathrm{Span}\{ E_i^* A_{i-1} v,\ E_{i-1}^* A_i v \mid 1 \le i \le D \} \\ &= \mathrm{Span}\{ v_i^+, v_{i-1}^- \mid 1 \le i \le D \} \\ &= \mathrm{Span}\{ v_1^+, v_2^+ \cdots, v_D^+, v_0^-, \cdots v_{D-1}^- \} \end{aligned}$$

by Lemma 62(v)(Lec 34-1)

But $v_0^- = E_0^* A_1 v = 0$ since $v \in (E_1^* V)_{new}$, and $v_1^- \in \mathrm{Span}\, v_1^+$.

Indeed,

$$v_1^- = E_1^* A_2 v = (-1 - a_0(Tv))\, v_1^+,$$

where $a_0(Tv)$ is the eigenvalue of v associated with $\tilde{A}$.

To see this observe

$$\begin{aligned} 0 &= \tilde{J} v \\ &= E_1^* \left(\sum_{i=0}^{D} A_i \right) E_1^* v \\ &= E_1^* \left(\sum_{i=0}^{2} A_i \right) E_1^* v \\ &= v + a_0(Tv) v + v_1^- \end{aligned}$$

Therefore

$$Tv = \mathrm{Span}\{ v_1^+, v_2^+, \ldots, v_D^+, v_2^-, \ldots, v_{D-1}^- \}$$

(iii) $v_i^+, v_i^- \in E_i^* V$

(iv) Suppose Tv is reducible, i.e., $Tv = W_1 + W_2$ (orthogonal direct sum of nonzero T-modules)

$E_1^* Tv = E_1^* W_1 + E_1^* W_2$ has dimension 1 by (iii)

Assume $v \in E_1^* W_1$, then $Tv \subseteq W_1$, a contradiction

LEMMA 64 (originally Lemma 61)

With the notation of Lemma 62 assume $0 \neq v \in (E_1^* V)_{new}$ is an eigenvector for $\tilde{A} := E_1^* A E_1^*$.

(i) Tv is thin $\Leftrightarrow$ $M^* v^* \leq M v$ (See remark below!)

(ii) Let W denote any irreducible T-module with endpoint 1. Then

$$W = Tv'$$

for some $0 \neq v' \in (E_1^* V)_{new}$ that is an eigenvector of $\tilde{A}$.

(iii) Denote eigenvalue of $\tilde{A}$ associated to v (resp. v') by $a_0(Tv)$ (resp. $a_0(Tv')$).

Then Tv, Tv' are isomorphic T-module $\Leftrightarrow$ $a_0(Tv) = a_0(Tv')$.

(iv) $E_1^* T E_1^*$ has basis

$$\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2, \dots, \tilde{A}^{\ell-1}$$

where $\ell = \#$ of mutually nonisomorphic T-modules with endpoint 1.

Proof.

(i) If Tv is thin, then by Lemma 15 (lec 9-1) $Tv = Mv$. Hence $M^* v^* \subseteq M v$.

$\boxed{\text{HS}}$ Note: originally this was Tv is thin $\Leftrightarrow$ $M^* v^* = M v$. This is not the case in general. Suppose Γ is thin. Let W be an irreducible T-module of endpoint 1. $k \leq m$

Then $W \cap E_1^* V \ni v \neq 0 \rightarrow v^* \in W \cap E_1 V$ gives one to one and

$J(v,d)$: $v \geq 2d$ BCN 255-256

$b_j = (d-j)(v-d-j)$, $c_j = j^2$

$\theta_j = (d-j)(v-d-j) - j$ $m_j = \binom{v}{j} - \binom{v}{j-1}$.

In particular

$k = b_0 = d(v-d) > m_1 = v-1$ if $d \geq 2$

and $J(v,d)$ is thin. Lec 8-5.

So $|X| E_1 v = v^*$ may be 0 sometimes.

But as Tv is dual thin of diameter at least $D-2$ (Lec 14-4), the dual end point $r^* \leq 2$. so in that case $E_2 v \neq 0$. Hence if $D \geq 3$, $E_2 v \neq 0$ always.

HS Now assume $M^* v^* \subseteq Mv. = Tv$

$Mv = \{ f(A)v \mid f(\lambda) \in \mathbb{C}[\lambda] \}$

So $E_i Tv = E_i Mv \in \mathrm{Span}(E_i v)$.

Hence Tv is dual thin.

Now we can construct a basis

$0 \neq w_0^* \in E_{r^*}^* W$ r^*: dual endpoint

$w_0^*, w_1^*, \ldots, w_d^* \in W = Tv$

where $w_i^* = E_{r^*+i}^* A_1^{*\,i} w_0^*$

$A_1^* w_i^* = w_{i+1}^* + a_i^* w_i^* + x_i^* w_{i-1}^*$ and $w_i^* = p_i^*(A^*) w_0^*$.

$E_{r^*+i}^* A_1^* E_{r^*+i}^* \mid_{E_{r^*+i}^* W} = a_i^* \cdot 1 \mid_{E_{r^*+i}^* W}$

$E_{r^*+i-1}^* A^* E_{r^*+i}^* A^* E_{r^*+i-1}^* \Big|_{E_{r^*+i-1}^* W} = x_i^* 1 \Big|_{E_{r^*+i-1}^* W}$

See Lemma 15 (Lec 9-1) and Lemma 41 (Lec 22-4)

From above $Tv = M^* w_0^*$

So $E_i^* Tv = E_i^* M^* w_0^* \in \mathrm{Span}\{ E_i^* w_0^* \}$

Thus Tv is thin.

※ Need to write down the dual at least for Lemma 15, Cor 16.

(ii) $E_1^* W$ is an $\tilde{A}$-module. So there exists $0 \neq v' \in E_1^* W$ that is an eigenvalue for $\tilde{A}$.
Also $Tv' \subseteq W$.
Since W is irreducible, $Tv' = W$.

(iii) Suppose $\sigma : Tv \to Tv'$ is an isomorphism of T-modules.
Recall $\sigma s = s \sigma$ for $\forall s \in T$.

$$\begin{aligned} \mathrm{Span}\{\sigma v\} &= \sigma E_1^* T v \\ &= E_1^* \sigma T v \\ &= E_1^* T v' \\ &= \mathrm{Span}\{v'\} \end{aligned}$$

Hence

$$\sigma \tilde{A} v = \sigma (a_0(Tv) v) = a_0(Tv) \sigma v$$
$$\shortparallel$$
$$\tilde{A} \sigma v = a_0(Tv') \sigma v$$

Since $\sigma v \neq 0$,

$$a_0(Tv) = a_0(Tv').$$

Now suppose $a_0(Tv) = a_0(Tv')$.
Show

$$\begin{aligned} \sigma : Tv &\to Tv' \\ sv &\to sv' \end{aligned} \qquad (s \in T)$$

is an isomorphism of T-modules.

Pick $s \in T$. Require

$$sv = 0 \leftrightarrow sv' = 0$$

WLOG, $s \in TE_1^*$,
since $v, v' \in E_1^* V$.

Now

$$0 = sv \leftrightarrow 0 = \| sv \|^2 = \bar{v}^t \bar{s}^t s v$$

But $\bar{s}^t s \in E_1^* T E_1^*$

Hence by LEMMA 61 (iii) (Lec33-7)

$$\bar{s}^t s = \alpha \tilde{J} + p(\tilde{A})$$

for some $\alpha \in \mathbb{C}$ and $p(\lambda) \in \mathbb{C}[\lambda]$.

Thus

$$0 = \| sv \|^2 = \bar{v}^t (\alpha \tilde{J} + p(\tilde{A})) v$$

$$= \| v \|^2 p(a_0(Tv)), \quad (\text{since } \tilde{J} v = 0).$$

$$\leftrightarrow \quad 0 = p(a_0(Tv)).$$

Replacing v by v', we have

$$0 = sv' \leftrightarrow 0 = p(a_0(Tv')).$$

$$\leftrightarrow 0 = p(a_0(Tv))$$

$$\leftrightarrow 0 = sv$$

as desired.

(iv) ℓ = # of mutually nonisomorphic T-modules with end point 1.

= # of distinct eigenvalues of $\tilde{A} : (E_1^* V)_{new} \to (E_1^* V)_{new}$

= the degree of minimal polynomial of

$$\tilde{A} : (E_1^* V)_{new} \to (E_1^* V)_{new}.$$

Claim 1 $\tilde{J}, E_1^*, \tilde{A}, \cdots, \tilde{A}^{\ell-1}$ are linearly independent.

(proof). Suppose not. Then

$$\alpha \tilde{J} + p(\tilde{A}) = 0$$

for some $\alpha \in \mathbb{C}$ and $p(\lambda) \in \mathbb{C}[\lambda]$ with $\deg p \leq \ell - 1$.

But $\tilde{J}|_{(E_1^* V)_{new}} = 0$ implies

$$p(\tilde{A})|_{(E_1^* V)_{new}} = 0$$

Since

$\deg p <$ deg of minimal polynomial of $\tilde{A}|_{(E_1^* V)_{new}}$.

we find p is identically 0.

Then α is identically 0 also.

Claim 2 $\tilde{J}, E_1^*, \tilde{A}, \cdots, \tilde{A}^{\ell-1}$ span $E_1^* T E_1^*$.

(Proof) It needs to show

$\tilde{J}, E_1^*, \tilde{A}, \cdots, \tilde{A}^{\ell}$ are linearly dependent (*)

Let m denote the minimal polynomial of $\tilde{A}|_{(E_1^* V)_{new}}$. So $m(\tilde{A})(E_1^* V)_{new} = 0$.

Observe :

$$E_1^* V = (E_1^* V)_{new} + \mathrm{Span}\{ A\hat{x} \}$$

(direct sum of $E_1^* T E_1^*$ - modules)

$$m(\tilde{A}) A\hat{x} = f \cdot A\hat{x} \qquad \text{for some } f \in \mathbb{C}.$$

On the other hand

$$\tilde{J} A\hat{x} = k A\hat{x} , \qquad (k : \text{valency of } \Gamma)$$

Therefore

$$m(\tilde{A}) - \frac{f}{k} \tilde{J} = 0 ,$$

and (*) holds.

Lecture 35 Mon. April 26, 1993

Theorem 65 (originally Theorem 62)
Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$.
Assume Γ is Q-polynomial w.r.t.
$E_0, E_1, \cdots, E_D$.
Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$.
(i) Up to isomorphism, there are at most 4 thin, irreducible T-modules with endpoint 1.
(ii) Suppose Γ is thin w.r.t. x. Then
$$\dim E_1^* T E_1^* \leq 5.$$

Proof.
(ii) is immediate from (i) and part (iv) of LEMMA 64 (Lec 34-7).

Proof of (i).
<u>Claim 1.</u> $E_1^* M E_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A})$

<u>Proof of Claim 1</u>
$$E_1^* M E_1^* = \mathrm{Span}\{E_1^*, E_1^* A E_1^*, E_1^* A_2 E_1^*, E_1^* A_3 E_1^*, \cdots\}$$
But $E_1^* A_h E_1^* = 0$ if $h > 2$
(by LEMMA 35 Lec 20-3)
So $E_1^* M E_1^* = \mathrm{Span}\{E_1^*, E_1^* A E_1^*, E_1^* A_2 E_1^*\}$

Also
$$\begin{aligned}\tilde{J} &= E_1^* J E_1^* \\ &= E_1^* \left(\sum_{h=0}^{D} A_h\right) E_1^* \\ &= E_1^* + E_1^* A E_1^* + E_1^* A_2 E_1^* \qquad \text{(See Lec 33-6)}\end{aligned}$$

So $E_1^* M E_1^* = \mathrm{Span}\{E_1^*, E_1^* A E_1^*, \tilde{J}\}$
We are done since $\tilde{A} = E_1^* A E_1^*$.

Claim 2 $\quad E_1^* M M^* M E_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2)$

Proof of Claim 2 $\quad \supseteq$: clear

$\subseteq$: In LEMMA 61 (i) (Lec33-7), we saw

$$E_1^* T = E_1^* E_0 M^* + E_1^* M + E_1^* E_1 M^* + E_1^* E_1 E_1^* M + \cdots$$

In fact the proof of that Lemma gives a sequence,

$$E_1^* M M^* = E_1^* E_0 M^* + E_1^* M + E_1^* E_1 M^*$$

$$E_1^* M M^* M = E_1^* E_0 M^* + E_1^* M + E_1^* E_1 M^* + E_1^* E_1 E_1^* M \quad -(1)$$

$$E_1^* M M^* M M^* = E_1^* E_0 M^* + E_1^* M + E_1^* E_1 M^* + E_1^* E_1 E_1^* M + E_1^* E_1 E_1^* E_1 M^*$$

$$\vdots$$

Multiply (1) through on the right by E_1^* to get

$$E_1^* M M^* M E_1^* = E_1^* M E_1^* + E_1^* E_1 E_1^* M E_1^*$$
$$= \mathrm{Span}\{\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2\},$$

since $\tilde{J}^2, \tilde{A}\tilde{J} = \tilde{J}\tilde{A} \in \mathrm{Span}\{\tilde{J}\}$,

This proves Claim 2.

Now let W denote any irreducible T-module with endpoint 1, and pick $0 \neq v \in E_1^* W$.

Set $\quad v_i^+ = E_i^* A_{i-1} E_1^* v, \quad v_i^- = E_i^* A_{i+1} E_1^* v \quad (1 \leq i \leq D)$

We know by Lemma 63 (ii) (Lec34-4) that

W is thin $\leftrightarrow$ v_i^+, v_i^- are linearly dependent $\forall i$ $(2 \leq i \leq D-1)$

In general

$$\Phi_i = \det \begin{pmatrix} \|v_i^+\|^2 & \langle v_i^+, v_i^- \rangle \\ \langle v_i^+, v_i^- \rangle & \|v_i^-\|^2 \end{pmatrix} \geq 0$$

with equality iff v_i^+, v_i^- are linear dependent.
(because Φ_i is a Gram matrix.)

Fix an integer i $(2 \leq i \leq D-1)$.

<u>Claim 3</u> There exists $p^{++} \in \mathbb{C}[\lambda]$, $\deg p^{++} \leq 2$ that depends only on the intersection numbers s.t. $\|v_i^+\|^2 = \|v\|^2\, p^{++}(a_0(W))$

<u>Proof of Claim 3</u>

$$\|v_i^+\|^2 = \bar{v}^t E_1^* A_{i-1} E_i^* E_i^* A_{i-1} E_1^* v$$

$$= \bar{v}^t E_1^* A_{i-1} E_i^* A_{i-1} E_1^* v.$$

But

$$E_1^* A_{i-1} E_i^* A_{i-1} E_1^* \in E_1^* M M^* M E_1^*$$

$$= \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2)$$

by Claim 2.

So $\exists \alpha \in \mathbb{C}$, $\exists p^{++} \in \mathbb{C}[\lambda]$ $\deg p^{++} \leq 2$.
s.t. $E_1^* A_{i-1} E_i^* A_{i-1} E_1^* = \alpha \tilde{J} + p^{++}(\tilde{A})$ $\quad (\tilde{A}^0 = E_1^*)$

Now $\|v_i^+\|^2 = \bar{v}^t(\alpha \tilde{J} + p^{++}(\tilde{A}))v$

$$= \|v\|^2\, p^{++}(a_0(W)),$$

since $\tilde{J}v = 0$ and $\tilde{A}v = a_0(W)\,v$.

This proves Claim 3.

($\boxed{\text{HS}}$ Proof of Claim 1, 2 together with those of Lem 58, 59, 60, 61 shows that these alternations can be written by intersection

Similarly $\exists\ p^{--},\ p^{+-} \in \mathbb{C}[\lambda]$, $\deg p^{--},\ \deg p^{+-} \leq 2$

s.t.
$$\|v_i^-\|^2 = \|v\|^2 p^{--}(a_0(W))$$
$$\langle v_i^+, v_i^- \rangle = \|v\|^2 p^{+-}(a_0(W))$$

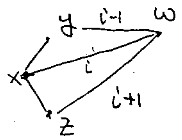
<u>Claim 4</u> $\quad E_1^* A_{i-1} E_i^* A_{i+1} E_1^* = (\tilde{J} - \tilde{A} - E_1^*)\, p^2_{i-1,\, i+1}$.

In particular, $\quad p^{+-}(\lambda) = -\, p^2_{i-1\ i+1}(\lambda + 1)$.

<u>Proof of Claim 4</u>

Pick $y, z \in X$ s.t. $\partial(x,y) = \partial(x,z) = 1$.

$$(\mathrm{LHS})_{yz} = \sum_{w \in X} (E_1^* A_{i-1} E_i^*)_{yw} (E_i^* A_{i+1} E_1^*)_{wz}$$
$$= \sum_{w\in X,\ \partial(y,w) = i-1,\ \partial(x,w) = i,\ \partial(w,z) = i+1} 1$$
$$= \begin{cases} 0 & \text{if } \partial(y,z) = 0 \\ 0 & \text{if } \partial(y,z) = 1 \\ p^2_{i-1\ i+1} & \text{if } \partial(y,z) = 2 \end{cases}$$



$$= (\mathrm{RHS})_{yz}$$

Note that $\quad E_1^* A_2 E_1^* = \tilde{J} - \tilde{A} - E_1^*$.

Now
$$\langle v_i^+, v_i^- \rangle = \bar{v}^t E_1^* A_{i-1} E_i^* A_{i+1} E_1^* v$$
$$= p^2_{i-1\, i+1} \left(\bar{v}^t (\tilde{J} - \hat{A} - E_1^*) v \right)$$
$$= -(a_0(W) + 1)\, p^2_{i-1,\, i+1} \|v\|^2$$

<u>Claim 5</u> $\deg p^{++} = \deg p^{--} = 2.$?

(only need for some i).

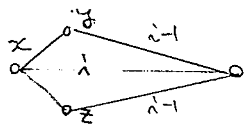
<u>Proof of Claim 5.</u>

We need to calculate p^{++}, p^{--}.

$\boxed{HS}$ Pick $y, z \in X$ s.t. $\partial(x,y) = \partial(x,z) = 1$

$(E_1^* A_{i-1} E_i^* A_{i-1} E_1^*)_{yz}$

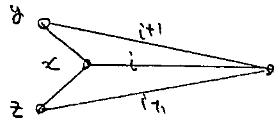
$= |\Gamma_{i-1}(y) \cap \Gamma_i(x) \cap \Gamma_{i-1}(z)|$



$= \begin{cases} p^1_{i-1\, i} & \text{if } \partial(y,z)=0 \\ \cdots & \\ \cdots & \end{cases}$

$(E_1^* A_{i+1} E_i^* A_{i+1} E_1)^*_{yz}$

$= |\Gamma_{i+1}(y) \cap \Gamma_i(x) \cap \Gamma_{i+1}(z)|$



$= \begin{cases} p^1_{i+1\, i} & \text{if } \partial(y,z)=0. \\ \cdots & \\ \cdots & \end{cases}$

- $E_1^* A E_2^* A E_1^* = (c_2-1)\tilde{J} + (b_0 - c_2)E_1^* + (a_1 - c_2)\tilde{A} - \tilde{A}^2$
- $E_1^* A_3 E_2^* A_3 E_1^*$

Conclusion

$$\Phi_i = \det \begin{pmatrix} \|v_i^+\|^2 & \langle v_i^+, v_i^- \rangle \\ \langle v_i^+, v_i^- \rangle & \|v_i^-\|^2 \end{pmatrix}$$

$$= \|v\|^4 \left(p^{++}(\lambda)\, p^{--}(\lambda) - (p^2_{i-1\, i+1})^2 (\lambda+1)^2 \right)$$

$$\geq 0 \qquad\qquad (\lambda = a_0(W))$$

W : thin $\Leftrightarrow$ $\Phi_i(\lambda) = 0$ $\quad \forall i \quad (2 \leq i \leq D-1)$

Each Φ_i is degree 4 so at most 4 solutions for λ.
Since λ determines the isomorphism class of W by LEMMA 64. (iii),
there are at most 4 different thin irreducible modules W of endpoint 1 up to isomorphism.

Note : In fact $\Phi_i(\lambda)$ is independent of i up to scalar multiple $\quad (2 \leq i \leq D-1)$.
If Γ has classical parameters (q, D, α, β), the roots are

$$\beta-\alpha-1, \quad -1, \quad -q-1, \quad \alpha q \frac{(q^{D-1}-1)}{q-1} - 1$$

$$A_i^* = \sum_{j=0}^{D} q_i(j)\, E_j^* \qquad E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j$$

$$E_i^* = \frac{1}{|X|} \sum_{j=0}^{D} p_i(j)\, A_j^* \qquad A_i = \sum_{j=0}^{D} p_i(j) E_j$$

$$A_0^* = I$$

$$\sum_{j=0}^{D} A_j^* = |X| E_0^* \qquad E_1 = \frac{1}{|X|} \sum_{j=0}^{D} \theta_j^* A_j$$

$$E_1^* A E_2^* + E_1^* A E_1^* + E_1^* A E_0^* = E_1^* A \sum_{j=0}^{D} E_j^* = E_1^* A$$

- $E_1^* A E_2^* = E_1^* A - E_1^* A E_0^* - E_1^* A E_1^*$

- $$E_1^* A_2 E_1^* = |X| E_1^* E_0 E_1^* - E_1^* A_0 E_1^* - E_1^* A E_1^*$$
  $$= \tilde{J} - E_1^* - \tilde{A}$$

- $E_1^* A E_2^* A E_1^*$:

$$= (E_1^* A - E_1^* A E_0^* - E_1^* A E_1^*)(A E_1^* - E_0^* A E_1^* - E_1^* A E_1^*)$$

$$= E_1^* A^2 E_1^* - E_1^* A E_0^* A E_1^* - E_1^* A E_1^* A E_1^*$$
$$- \cancel{E_1^* A E_0^*} A E_1^* + \cancel{E_1^* A E_0^*} A E_1^* - \cancel{E_1^* A E_1^*} A E_1^* + \cancel{E_1^* A E_1^*} A E_1^*$$

$$= b_0 E_1^* + a_1 E_1^* A E_1^* + c_2 E_1^* A_2 E_1^* - \tilde{J} - \tilde{A}^2$$

$$= b_0 E_1^* + a_1 \tilde{A} + c_2 \tilde{J} - c_2 E_1^* - c_2 \tilde{A} - \tilde{J} - \tilde{A}^2$$

$$= (c_2 - 1)\tilde{J} + (b_0 - c_2) E_1^* + (a_1 - c_2)\tilde{A} - \tilde{A}^2$$

- $E_1^* E_1 E_1^* = \frac{1}{|X|}\left((\theta_0^* - \theta_2^*) E_1^* + (\theta_1^* - \theta_2^*)\tilde{A} + \theta_2^* \tilde{J}\right)$ (Lemma 60 Lec33-6)

- $E_1^* E_1 E_0^* E_1 E_1^* = \frac{\theta_1^{*2}}{|X|^2} \tilde{J}$ $\qquad E_1 = \frac{1}{|X|} \sum_{j=0}^{D} \theta_j^* A_j$

$$\cdot\ E_1^* E_1 E_1^* E E_1^* = \frac{1}{|X|^2}\left((\theta_0^*-\theta_2^*)E_1^* + (\theta_1^*-\theta_2^*)\tilde{A} + \theta_2^*\tilde{J}\right)^2$$

$$= \frac{1}{|X|^2}\Big((\theta_0^*-\theta_2^*)^2 E_1^* + (\theta_1^*-\theta_2^*)^2\tilde{A}^2 + \theta_2^{*2}\tilde{J}^2$$

$$+2(\theta_0^*-\theta_2^*)(\theta_1^*-\theta_2^*)\tilde{A} + 2(\theta_0^*-\theta_2^*)\theta_2^*\tilde{J} + 2(\theta_1^*-\theta_2^*)\theta_2^*\tilde{A}\tilde{J}\Big)$$

$$= \frac{1}{|X|^2}\Big((b_0\theta_2^{*2} + 2(\theta_0^*-\theta_2^*)\theta_2^* + 2a_1(\theta_1^*-\theta_2^*)\theta_2^*)\tilde{J}$$

$$+(\theta_0^*-\theta_2^*)^2 E_1^* + 2(\theta_0^*-\theta_2^*)(\theta_1^*-\theta_2^*)\tilde{A} + (\theta_1^*-\theta_2^*)^2\tilde{A}^2\Big)$$

$$\begin{cases} \theta_3^* E_1^* A_3 E_2^* + \theta_2^* E_1^* A_2 E_2^* + \theta_1^* E_1^* A E_2^* = |X| E_1^* E_1 E_2^* \\ E_1^* A_3 E_2^* + E_1^* A_2 E_2^* + E_1^* A E_2^* = |X| E_1^* E_0 E_2^* \end{cases}$$

$$\cdot\ E_1^* A_3 E_2^* = \frac{1}{\theta_3^*-\theta_2^*}\left(|X| E_1^* E_1 E_2^* - |X|\theta_2^* E_1^* E_0 E_2^* + (\theta_2^*-\theta_1^*) E_1^* A E_2^*\right)$$

$$\begin{cases} \theta_2 E_1 A_2^* E_1 + \theta_1 E_1 A_1^* E_1 + \theta_0 E_1 = |X| E_1 E_1^* E_1 \\ E_1 A_2^* E_1 + E_1 A_1^* E_1 + E_1 = |X| E_1 E_0^* E_1 \end{cases}$$

$$\cdot\ E_1 A_2^* E_1 = \frac{1}{\theta_2-\theta_1}\left(|X| E_1 E_1^* E_1 - |X|\theta_1 E_1 E_0^* E_1 + (\theta_1-\theta_0) E_1\right)$$

$$\cdot\ E_1 A_1^* E_1 = \frac{1}{\theta_1-\theta_2}\left(|X| E_1 E_1^* E_1 - |X|\theta_2 E_1 E_0^* E_1 + (\theta_2-\theta_0) E_1\right)$$

$$\cdot\ E_1 E_2^* E_1 = \frac{1}{|X|}\left(p_2(2) E_1 A_2^* E_1 + p_2(1) E_1 A_1^* E_1 + p_2(0) E_1\right)$$

$$= \frac{1}{|X|}\Big(p_2(0) E_1 + \frac{p_2(1)}{\theta_1-\theta_2}\left(|X| E_1 E_1^* E_1 - |X|\theta_2 E_1 E_0^* E_1 + (\theta_2-\theta_0) E_1\right)$$

$$+ \frac{p_2(2)}{\theta_2-\theta_1}\left(|X| E_1 E_1^* E_1 - |X|\theta_1 E_1 E_0^* E_1 + (\theta_1-\theta_0) E_1\right)\Big)$$

$$= \frac{1}{|X|(\theta_2-\theta_1)}\Big(\left(p_2(0)(\theta_2-\theta_1) - p_2(1)(\theta_2-\theta_0) + p_2(2)(\theta_1-\theta_0)\right) E_1$$

$$+ |X|(p_2(2)-p_2(1)) E_1 E_1^* E_1 + |X|(\theta_2 p_2(1) - \theta_1 p_2(2)) E_1 E_0^* E_1$$

$\bullet$ $E_1^*E_1E_2^*E_1E_1^*$

$$= \frac{1}{|X|(\theta_2-\theta_1)}\left(p_2(0)(\theta_2-\theta_1)-p_2(1)(\theta_2-\theta_0)+p_2(2)(\theta_1-\theta_0)\right)E_1^*E_1E_1^*$$

$$+\frac{p_2(2)-p_2(1)}{\theta_2-\theta_1}E_1^*E_1E_1^*E_1E_1^* + \frac{\theta_2p_2(1)-\theta_1p_2(2)}{\theta_2-\theta_1}E_1^*E_1E_0^*E_1E_1^*$$

$$= \frac{1}{|X|^2(\theta_2-\theta_1)}\left(p_2(0)(\theta_2-\theta_1)-p_2(1)(\theta_2-\theta_0)+p_2(2)(\theta_1-\theta_0)\right)\left((\theta_0^*-\theta_2^*)E_1^*+(\theta_1^*-\theta_2^*)\tilde{A}+\theta_2^*\tilde{J}\right)$$

$$+\frac{p_2(2)-p_2(1)}{|X|^2(\theta_2-\theta_1)}\Big(\left(b_0\theta_2^{*2}+2(\theta_0^*-\theta_2^*)\theta_2^*+2a_1(\theta_1^*-\theta_2^*)\theta_2^*\right)\tilde{J}+(\theta_0^*-\theta_2^*)^2E_1^*$$

$$+2(\theta_0^*-\theta_2^*)(\theta_1^*-\theta_2^*)\tilde{A}+(\theta_1^*-\theta_2^*)^2\tilde{A}^2\Big)$$

$$+\frac{\theta_2p_2(1)-\theta_1p_2(2)}{|X|^2(\theta_2-\theta_1)}\theta_1^{*2}\tilde{J}$$

$$= \frac{1}{|X|^2(\theta_2-\theta_1)}\Big[\Big\{\left(p_2(0)(\theta_2-\theta_1)-p_2(1)(\theta_2-\theta_0)+p_2(2)(\theta_1-\theta_0)\right)\theta_2^* + \left(\theta_2p_2(1)-\theta_1p_2(2)\right)\theta_1^{*2}$$

$$+\left(p_2(2)-p_2(1)\right)\left(b_0\theta_2^{*2}+2(\theta_0^*-\theta_2^*)\theta_2^*+2a_1(\theta_1^*-\theta_2^*)\theta_2^*\right)\Big\}\tilde{J}$$

$$+\Big\{\left(p_2(0)(\theta_2-\theta_1)-p_2(1)(\theta_2-\theta_0)+p_2(2)(\theta_1-\theta_0)\right)(\theta_0^*-\theta_2^*)+\left(p_2(2)-p_2(1)\right)(\theta_0^*-\theta_2^*)^2\Big\}E_1^*$$

$$+\Big\{\left(p_2(0)(\theta_2-\theta_1)-p_2(1)(\theta_2-\theta_0)+p_2(2)(\theta_1-\theta_0)\right)(\theta_1^*-\theta_2^*)+2\left(p_2(2)-p_2(1)\right)(\theta_0^*-\theta_2^*)(\theta_1^*-\theta_2^*)\Big\}\tilde{A}$$

$$+\Big\{\left(p_2(2)-p_2(1)\right)(\theta_1^*-\theta_2^*)^2\Big\}\tilde{A}^2\Big]$$

$\bullet$ $$E_1^*E_1E_2^*E_0E_1^* = E_1^*E_1\left(\frac{1}{|X|}\sum_{j=0}^{D}p_2(j)A_j^*\right)E_0E_1^*$$

$$= \frac{p_2(1)}{|X|}E_1^*E_1A_1^*E_0E_1^* = \frac{p_2(1)}{|X|}E_1^*A_1^*E_0E_1^*$$

$$= \frac{p_2(1)\theta_1^*}{|X|}E_1^*E_0E_1^* = \frac{p_2(1)\theta_1^*}{|X|^2}\tilde{J}$$

$\bullet$ $$E_1^*E_1E_2^*AE_1^* = E_1^*E_1\left(AE_1^*-E_0^*AE_1^*-E_1^*AE_1^*\right)$$

$$= \theta_1E_1^*E_1E_1^* - |X|E_1^*E_1E_0^*E_0E_1^* - E_1^*E_1E_1^*AE_1^*$$

$$= \frac{1}{|X|}\left((\theta_0^*-\theta_2^*)E_1^*+(\theta_1^*-\theta_2^*)\tilde{A}+\theta_2^*\tilde{J}\right)\left(\theta_1E_1^*-\tilde{A}\right) - \theta_1^*\tilde{J}$$

$$= \frac{1}{|X|}\Big((\theta_2^*\theta_1-\theta_2^*a_1-|X|\theta_1^*)\tilde{J} + (\theta_0^*-\theta_2^*)\theta_1E_1^* + \left((\theta_1^*-\theta_2^*)\theta_1-(\theta_0^*-\theta_2^*)\right)\tilde{A} - (\theta_1^*-\theta_2^*)\tilde{A}^2\Big)$$

• $E_1^* A_3 E_2^* A_3 E_1^*$

$$= \frac{1}{(\theta_3^*-\theta_2^*)^2}\Big(|X| E_1^* E_1 E_2^* - |X|\theta_2^* E_1^* E_0 E_2^* + (\theta_2^*-\theta_1^*) E_1^* A E_2^*\Big)$$
$$\cdot \Big(|X| E_2^* E_1 E_1^* - |X|\theta_2^* E_2^* E_0 E_1^* + (\theta_2^*-\theta_1^*) E_2^* A E_1^*\Big)$$

$$= \frac{1}{(\theta_3^*-\theta_2^*)^2}\Big[|X|^2 E_1^* E_1 E_2^* E_1 E_1^* + |X|^2 \theta_2^{*2} E_1^* E_0 E_2^* E_0 E_1^* + (\theta_2^*-\theta_1^*)^2 E_1^* A E_2^* A E_1^*$$
$$-2|X|^2\theta_2^* E_1^* E_1 E_2^* E_0 E_1^* + 2|X|(\theta_2^*-\theta_1^*) E_1^* E_1 E_2^* A E_1^*$$
$$-2|X|\theta_2^*(\theta_2^*-\theta_1^*) E_1^* E_0 E_2^* A E_1^* \Big]$$

$$= \frac{1}{(\theta_3^*-\theta_2^*)^2}\Big[|X|^2 E_1^* E_1 E_2^* E_1 E_1^* + \theta_2^{*2} p_2(0)\tilde{J} + (\theta_2^*-\theta_1^*)^2 E_1^* A E_2^* A E_1^*$$
$$-2p_2(1)\theta_1^*\theta_2^*\tilde{J} - 2b_1\theta_2^*(\theta_2^*-\theta_1^*)\tilde{J}$$
$$+2(\theta_2^*-\theta_1^*)\Big((\theta_2^*\theta_1 - a_1\theta_2^* - |X|\theta_1^*)\tilde{J} + \theta_1(\theta_0^*-\theta_2^*)E_1^* + (\theta_1(\theta_1^*-\theta_2^*)-(\theta_0^*-\theta_2^*))\tilde{A}$$
$$-(\theta_1^*-\theta_2^*)\tilde{A}^2\Big)\Big]$$

(coefficient of $\tilde{A}^2$)

$$= \frac{1}{(\theta_3^*-\theta_2^*)^2}\Big[\frac{p_2(2)-p_2(1)}{\theta_2-\theta_1}(\theta_1^*-\theta_2^*)^2 - (\theta_2^*-\theta_1^*)^2 + 2(\theta_2^*-\theta_1^*)^2\Big]$$

$$= \frac{(\theta_2^*-\theta_1^*)^2}{(\theta_3^*-\theta_2^*)^2}\;\frac{p_2(2)-p_2(1)+(\theta_2-\theta_1)}{\theta_2-\theta_1} \qquad p_2(2)+p_1(2) \overset{?}{=} p_2(1)+p_1(1)$$

$$p_2(t) = \frac{1}{c_2}(t^2 - a_1 t - b_0) \qquad p_2(2)-p_2(1) = \frac{1}{c_2}\big(\theta_2^2-\theta_1^2 - a_1(\theta_2-\theta_1)\big)$$

$$\frac{p_2(2)-p_2(1)+\theta_2-\theta_1}{\theta_2-\theta_1} = \frac{1}{c_2}(\theta_2+\theta_1-a_1+c_2)$$

• $$E_1^* E_1 E_2^* A E_1^* = \frac{1}{|X|}\Big((\theta_2^*\theta_1 - a_1\theta_2^* - |X|\theta_1^*)\tilde{J} + \theta_1(\theta_0^*-\theta_2^*)E_1^*$$
$$+(\theta_1(\theta_1^*-\theta_2^*) - (\theta_0^*-\theta_2^*))\tilde{A} - (\theta_1^*-\theta_2^*)\tilde{A}^2\Big)$$

Lecture 36 Wed. April 28, 1993

Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$, Q-polynomial w.r.t. $E_0, E_1, \cdots, E_D$.
Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$.
Let W be an irreducible T-module of diameter d.
Recall that the endpoint

$$r(W) = \min\{ i \mid 0 \leq i \leq D,\ E_i^* W \neq 0 \}$$

DEF. The <u>dual endpoint</u> (w.r.t. above ordering $E_0, \cdots, E_D$)

$$r^*(W) = \min\{ i \mid 0 \leq i \leq D,\ E_i W \neq 0 \}$$

$$r(W) = 0 \Leftrightarrow r^*(W) = 0 \Leftrightarrow W : \text{trivial } T\text{-module}$$

(by Lemma 21 (Lec 11-1))

Suppose W is thin. Then W is dual thin.
(by Cor 16 (Lec 9-4))

Moreover,
$\{ i \mid E_i W \neq 0 \}$ is a subinterval of $\{0, 1, \cdots, D\}$.
(Same proof as for distance regular)

$\boxed{\text{HS}}$ LEMMA 9' (Lec 4-5) $A^* \equiv A_1^*(x)$.
W : irreducible T-module. $d^* = |\{ i \mid E_i W \neq 0 \}| - 1$

(i) $E_i A^* E_j \begin{cases} = 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases}$ $0 \leq i, j \leq D$

(ii) $A^* E_j W \subseteq E_{j-1} W + E_j W + E_{j+1} W$
$0 \leq j \leq D$ ($E_j = 0$ if $j < 0$ or $j > D$)

(iii) $E_j W \begin{cases} \neq 0 & \text{if } r^* \leq j \leq r^* + d^* \\ = 0 & \text{if } 0 \leq j < r^* \text{ or } r^* + d^* < j \leq D \end{cases}$

(iv) $E_i A^* E_j W \neq 0$ if $|i-j| = 1$ ($r^* \leq i, j \leq r^* + d^*$)

$\boxed{HS}$ continued.

Proof (i) Lemma 35 (Lec 19-8)

$$E_i A^* E_j = 0 \quad \leftrightarrow \quad q_{i1}^{j} = 0$$

Lemma 41 (Lec 22-4)

$$\Gamma: Q\text{-poly.} \leftrightarrow q_{i1}^{j} \begin{cases} = 0 & \text{if } |j-i|>1 \\ \neq 0 & \text{if } |j-i|=1 \end{cases}$$

$$\leftrightarrow E_i A^* E_j \begin{cases} = 0 & \text{if } |j-i|>1 \\ \neq 0 & \text{if } |j-i|=1 \end{cases}$$

(ii) $$A^* E_j W = \left(\sum_{i=0}^{D} E_i\right) A^* E_j W$$

$$= E_{j-1} A^* E_j W + E_j A^* E_j W + E_{j+1} A^* E_j W$$

$$\subseteq E_{j-1} W + E_j W + E_{j+1} W$$

(iii) Suppose $E_j W = 0$ for some j $(r^* \leq j \leq r^* + d^*)$

Then $r^* < j$ by definition of r^*

Set $\tilde{W} = E_{r^*} W + E_{r^*+1} W + \cdots + E_{j-1} W$

Then $\tilde{W}$ is T-inv. and $0 \subsetneq \tilde{W} \subsetneq W$.

A contradiction.

(iv) Suppose $E_{j+1} A^* E_j W = 0$ for some j $(r^* \leq j < r^* + d^*)$

Then $\tilde{W} = E_{r^*} W + \cdots + E_j W$ is T-inv.

If $E_{j-1} A^* E_j W = 0$ for some j $(r^* < j \leq r^* + d^*)$

then $\tilde{W} = E_j W + \cdots + E_{r^*+d^*}$ is T-inv

Moreover $0 \subsetneq \tilde{W} \subsetneq W$ in both cases.

A contradiction

DEF. Let W be an irreducible dual thin T-module with dual endpoint r^*, diameter d^*

Let $a_i^* = a_i^*(W) \in \mathbb{C}$ satisfying

$$E_{r^*+i} A^* E_{r^*+i} |_{E_{r^*+i} W} = a_i^* \cdot 1 |_{E_{r^*+i} W}$$

Let $x_i^* = x_i^*(W) \in \mathbb{C}$ satisfying

$$E_{r^*+i-1} A^* E_{r^*+i} A^* E_{r^*+i-1} |_{E_{r^*+i-1} W} = x_i^* 1 |_{E_{r^*+i-1} W}$$

$\boxed{\text{HS}}$ continued

LEMMA 15' (Lec 9-1) With above notation, the following hold

(i) $a_i^* \in \mathbb{R} \quad (0 \le i \le d^*)$

(ii) $x_i^* \in \mathbb{R}^{>0} \quad (1 \le i \le d^*)$

(iii) Pick $0 \ne w_0^* \in E_{r^*}W$. Set $w_i^* = E_{r^*+i}A^{*i}w_0^* \quad \forall i$

then (iii a) $w_0^*, w_1^*, \cdots, w_{d^*}^*$ is a basis for W, $w_{-1}^* = w_{d^*+1}^* = 0$.

(iii b) $A^*w_i^* = w_{i+1}^* + a_i^* w_i^* + x_i^* w_{i-1}^* \quad (0 \le i \le d^*)$

(iv) Define $p_0^*, p_1^*, \cdots, p_{d^*+1}^* \in \mathbb{R}[\lambda]$ by

$p_0^* = 1, \quad \lambda p_i^* = p_{i+1}^* + a_i^* p_i^* + x_i^* p_{i-1}^* \quad (0 \le i \le d^*)\ p_{-1}^* = 0$

then (iva) $p_i^*(A^*)w_0^* = w_i^* \quad (0 \le i \le d^*+1)$

(iv b) $p_{d^*+1}^*$ is the minimal polynomial of $A^*|W$.

Proof. (i) $A^* = \sum_{j=0}^{D} \theta_j^* E_j^* \qquad \theta_j^* = \mathfrak{z}_1(j) = |X|(E_1)_{xy} \in \mathbb{R}$, $\partial(x,y)=j$

a_i^* is an eigenvalue of a real symmetric matrix $E_{r+i}^* A^* E_{r+i}^*$

(ii) $B = E_{r+i}^* A^* E_{r^*+i-1}$

x_i^* is an eigenvalue of a real symmetric matrix B^tB.

Let $\mathrm{Span}\{v_{i-1}\} = E_{r^*+i-1}W$

and $Bv_{i-1} \ne 0$ by Lemma 9' (iv) for $1 \le i \le d^*$

So $x_i^* \in \mathbb{R}^{>0} \quad (1 \le i \le d^*)$

(iii a) Observe $w_i^* = E_{r^*+i}A^*E_{r^*+i-1}w_{i-1}^*$

So $w_i^* \ne 0 \quad (0 \le i \le d^*)$ by Lemma 9' (iv)

Hence $W = \mathrm{Span}(w_0^*, \cdots, w_{d^*}^*)$ by Lemma 9' (ii)

(iii b) $A^*w_i^* = E_{r^*+i+1}A^*w_i^* + E_{r+i}^*A^*w_i^* + E_{r^*+i-1}A^*w_i^*$

$= w_{i+1}^* + E_{r+i}^* A E_{r^*+i} w_i + E_{r^*+i-1}A^*E_{r^*+i}A^*E_{r^*+i-1}w_{i-1}^*$

$= w_{i+1}^* + a_i^* w_i^* + x_i^* w_{i-1}^*$

(iva) Clear for $i=0$. Assume OK for $0, \cdots, i$.

$p_{i+1}^*(A^*)w_0^* = (A^* - a_i^* I)w_i^* - x_i^* w_{i-1}^* = w_{i+1}^*$

(iv b) By definition $p_{d+1}^*(A^*)w_0^* = 0$

Since $W = \{p(A^*)w_0^* \mid p(\lambda) \in \mathbb{C}[\lambda]\}$

$p_{d+1}^*(A^*)W = 0$. and $p_{d+1}^*(\lambda)$ is the minimal poly.

as $w_0^*, \cdots, w_{d^*}^*$ is a basis

$\boxed{\text{HS}}$ continued.

Cor 16′ (Lec 9-4) With the notation above
Let W be a dualthin irreducible T-module with dual endpoint $r^*(W)$ and dual diameter d^*.
Then (i) W is thin
(ii) $d^* = d = |\{i \mid E_i^* W \neq 0\}| - 1$

Proof. Set as in Lemma 15′.
$w_i^* = p_i^*(A^*) w_0^* \in E_{r+i}^* W$
Then $w_0^*, w_1^*, \cdots, w_{d^*}$ is a basis for W.
We have $W = M^* w_0^*$
So $E_i^* W = E_i^* M^* w_0^* = \text{Span}(E_i^* w_0^*)$
Thus W is thin and so we have (ii).

$\boxed{\text{HS}}$ Suppose $r(W) = 1$ and $r^*(W) = 2$.
Then $E_1 v = 0$.
On the other hand

$$E_1^* E_1 E_1^* = \frac{1}{|X|}((\theta_0^* - \theta_2^*) E_1^* + (\theta_1^* - \theta_2^*)\tilde{A} + \theta_2^* \tilde{J})$$

Since $E_0 v = 0$, $\tilde{J} v = 0$. So

$$0 = E_1^* E_1 E_1^* v = \frac{1}{|X|}((\theta_0^* - \theta_2^*) E_1^* + (\theta_1^* - \theta_2^*)\tilde{A} + \theta_2^* \tilde{J}) v$$

$$= \frac{1}{|X|}(\theta_0^* - \theta_2^*) v + \frac{1}{|X|}(\theta_1^* - \theta_2^*) a_0(W) v.$$

We have
$$a_0(W) = -\frac{\theta_2^* - \theta_0^*}{\theta_2^* - \theta_1^*}$$

$\therefore$ $a_0(W)$ is uniquely determined

Suppose $r(W)=1$. Then $d(W)=D-2$ or $D-1$ by Lemma 28 (iii) (Lec 14-2) (See also Lemma 29 (Lec 14-4))

Case $d(W)=D-2$

Then $E_1W=0$ implies $r^*(W)=2$

$E_1W\neq 0$ implies $r^*(W)=1$

Case $d(W)=D-1$

Then $r^*(W)=1$.

Up to isomorphism

there are at most 3 thin irreducible T-modules $r(W)=1$ $r^*(W)=1$.

there are at most 1 thin irreducible T-modules $r(W)=1$, $r^*(W)=2$

there are none thin irreducible T-modules $r(W)=1$ $r^*(W)>2$.

By dual argument

there are at most 3 thin irreducible T-modules $r^*(W)=1$, $r(W)=1$

there are at most 1 thin irreducible T-modules $r^*(W)=1$, $r(W)=2$

there are none thin irreducible T-modules $r^*(W)=1$ $r(W)>2$

$\boxed{\text{HS}}$?

Conjecture Let $\Gamma = (X, E)$ be a thin distance regular graph of diameter $D \geq 3$.
Let E_1 be any primitive idempotent ($E_1 \neq E_0$).
Then the following are equivalent.

(i) For $\forall x \in X$, there is no irreducible T-module W with $r(W) > 2$ and $EW \neq 0$,
there exists at most 1 irreducible T-module W with $r(W) = 2$ and $EW \neq 0$
there exist at most 3 irreducible T-modules W with $r(W) = 1$ and $EW \neq 0$.

(ii) Γ is Q-polynomial w.r.t. E.

Conjecture Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$, Q-polynomial w.r.t $E_0, E_1, \cdots, E_D$.
Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$.
Let W denote an irreducible T-module with endpoint r, dual endpoint r^*, diameter d, dual diameter d^*.
Then

(i) $d = d^*$

(ii) there exists s ($r \leq s \leq r+d$) s.t

$$1 = \dim E_r^* W \leq \dim E_{r+1}^* W \leq \cdots \leq \dim E_s^* W \geq \cdots \geq \dim E_{r+d}^* W = 1$$

(iii) there exists s^* ($r^* \leq s^* \leq r^* + d^*$) s.t.

$$1 = \dim E_{r^*} W \leq \cdots \leq \dim E_{s^*} W \geq \cdots \geq \dim E_{r^*+d^*} W = 1$$

Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$.
Θ-polynomial w.r.t. $E_0, E_1, \ldots, E_D$
Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$
Let W denote irreducible T-module with endpoint 1

Conjecture : The following are equivalent.

(i) W is not thin.

(ii) The sequence $\dim E_1^* W, \dim E_2^* W, \ldots, \dim E_D^* W$ equals $1, 2, 2, \ldots, 2, 1$

(iii) $v, Av, A_2 v, \ldots, A_{D-2} v$,

$v^*, A^* v^*, A_2^* v^*, \ldots, A_{D-2}^* v^*$

is a basis for W, where $0 \neq v \in E_1^* W$ and

$v^* = |X| E_1 v$

(iv) $v_1^+, v_2^+, \ldots, v_D^+, v_2^-, v_3^-, \ldots, v_{D-1}^-$

is a basis for W, where

$$v_i^+ = E_i^* A_{i-1} v, \qquad v_i^- = E_i^* A_{i+1} v$$

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Problem Let B denote the orthogonal basis for W obtained by applying the Gram Schmidt procedure to the basis in (iv)
Find the matrix representing A w.r.t. this basis.
I believe the entries are nicely factorable expressions in the basic variables,

$q, s, s^*, r_1, r_2 \ldots$

(Hint : use Theorem)

If not, find some nice basis for W and find the matrices representing A, A^* wrt this basis.
Perhaps some orthogonal basis based on (iii).

Algebraically, everything is determined by the intersection numbers and $a_0(W)$.
Combinatorially, certain quantities must be nonnegative integers, Does this give some new bounds or other information on $a_0(W)$?

Lecture 37 Fri. April 30, 1993

LEMMA 66 (originally Lemma 63 continued)
Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$, Q-polynomial w.r.t. $E_0, E_1, \cdots, E_D$.
Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$.
Let W be an irreducible T-module of endpoint 1.
If $\dim E_2^* W = 1$, then W is thin.

Proof. Pick $0 \neq v \in E_1^* W$.

We want to show that

- $FR^i v \in \mathrm{Span}(R^i v)$ $\quad (0 \leq i \leq D-1)$
- $LR^i v \in \mathrm{Span}(R^{i-1} v)$ $\quad (1 \leq i \leq D-1)$

We have that

(1) $FR^2 E_j^* \in \mathrm{Span}(RFRE_j^*, R^2FE_j^*, R^2E_j^*)$ $\quad (0 \leq j \leq D-3)$

(2) $LR^2 E_j^* \in \mathrm{Span}(RLRE_j^*, R^2LE_j^*, F^2RE_j^*, FRFE_j^*$
$RF^2E_j^*, RFE_j^*, FRE_j^*, RE_j^*)$ $\quad (0 \leq j \leq D-3)$

by COROLLARY 53 (Lec 30-8)

<u>Claim</u> (a) $FR^i v \in \mathrm{Span}(R^i v)$ $\quad (0 \leq i \leq D-2)$
(b) $LR^i v \in \mathrm{Span}(R^{i-1} v)$ $\quad (1 \leq i \leq D-2)$

HS <u>Proof of Claim</u>

(a) By LEMMA 63 and our assumption,
$\dim E_1^* W = \dim E_2^* W = 1$.
So $Rv \neq 0$ ad $E_2^* W = \mathrm{Spa}(Rv)$.
We may assume $i \geq 2$. Then $R^{i-2} v \in E_{i-1}^* W$

$$FR^i v = FR^2 R^{i-2} v, \quad \text{if } i \leq D-2$$
$$\in R(FR + RF + R)R^{i-2} v$$

By induction
$$\in R(\mathrm{Span}(R^{i-1} v)) = \mathrm{Span}(R^i v)$$

$\boxed{\text{HS}}$ continued.

(b) If $i \le D-2$, then $R^{i-2}v \in E^*_{i-1}W$ with $i-1 \le D-3$.
Hence

$$LR^i v = LR^2(R^{i-2}v)$$
$$= (RLR + R^2L + F^2R + FRF + RF^2 + RF + FR + R)R^{i-2}v$$

By induction and (a)

$$\in \operatorname{Span}(R^{i-1}v)$$

Suppose $R^{D-1}v = 0$. Then
$\operatorname{Span}(v, Rv, \cdots, R^{D-2}v) = \tilde{W}$.
is invariant under M, M^*, hence under T.
Since W is irreducible, $W = \tilde{W}$ and
W is thin in this case.

Suppose $R^{D-1}v \neq 0$.

Observe: $v, Av, \cdots, A^{D-1}v \in \operatorname{Span}(v, Rv, \cdots, R^{D-1}v)$
Hence each
$R^i v$ is a polynomial of degree i in A
applied to v, and

$$\operatorname{Span}(v, Av, \cdots, A^{D-1}v)$$
$$= \operatorname{Span}(v, Rv, \cdots, R^{D-1}v)$$
$$= \operatorname{Span}(v, A_1v, \cdots, A_{D-1}v).$$

Also
$$A_D v = \underset{\overset{\shortparallel}{0}}{Jv} - \left(\sum_{h=0}^{D-1} A_h\right)v$$
$$\in \operatorname{Span}(v, A_1v, \cdots, A_{D-1}v)$$

Thus
$$Mv = \operatorname{Span}(v, Rv, \cdots, R^{D-1}v).$$

Therefore.
$$\operatorname{Span}(v, Rv, \cdots, R^{D-1}v) = \tilde{W}$$
is invariant under M, M^*, T. We have $W = \tilde{W}$ and W is thin.

DEF. Let $\Gamma = (X, E)$ be any regular graph (not necessarily connected.)
Let A be the adjacency matrix of Γ and J be the all 1's matrix.
Pick $0 \neq B \in \mathrm{Mat}_X(\mathbb{C})$.
B is a generalized adjacency matrix, if
(i) $\forall x, y \in X$: $B_{xy} \neq 0 \rightarrow A_{xy} \neq 0$ or $x = y$.
(ii) B is in the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, J.

Example. Any nonzero matrix of form
$\alpha A + \beta I$ $(\alpha, \beta \in \mathbb{C})$
is a generalized adjacency matrix.
If Γ is distance regular, all generalized adjacency matrices are of this form.

Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$. Assume Γ is thin and Q-polynomial.
Pick $x \in X$, and write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$.
Then

$$E_1^* T E_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2, \tilde{A}^3)$$

and $\dim E_1^* T E_1^* \leq 5$.
We will produce a "nice" spanning set

$$E_1^* T E_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, A^+ (= R^{-1} E_2^* A E_1^*), A^+ \tilde{A})$$

LEMMA 67 (originally Lemma 64)

Let $\Gamma = (X, E)$ be a thin distance-regular graph of diameter $D \geq 4$.
Fix $x \in X$, and write $E_i^* \equiv E_i^*(x)$, $R \equiv R(x)$.
Let Γ_1 denote the vertex subgraph induced on the 1^{st} subconstituent of Γ relative to x.
Then

$$\Delta = (R^{-1})^{i-1} E_i^* A_i E_1^* \qquad \text{--- (1)}$$

is a generalized adjacency matrix for Γ_1.
for $\forall i$ $(1 \leq i \leq D-3)$.

Proof.

Write $T \equiv T(x)$. Fix i $(1 \leq i \leq D-3)$.
Recall $R^{-1} \in T$ by LEMMA 54 (iv) (Lec 31-1)
Since $E_{i-1}^* R^{-1} E_i^* = R^{-1} E_i^*$ (by LEMMA 54 (ii) (Lec 31-1))

$$\Delta \in E_1^* T E_1^*$$

$$= \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2, \cdots)$$

by LEMMA 64 (iv) (Lec 34-7).

Hence Δ satisfies the condition (ii) of DEF in Lec 37-3.

To show (i), pick $y, z \in X$
s.t. $\partial(x, y) = \partial(x, z) = 1$, $\partial(y, z) = 2$.
We need to show

$$\Delta_{yz} = 0.$$

Suppose $\Delta_{yz} \neq 0$. Then

$$\langle \Delta \hat{y}, \hat{z} \rangle \neq 0.$$

<u>We will show this cannot occur.</u>

Notation : Set

$$E^*_{ij} = E^*_i(x)\, E^*_j(y) \qquad (0 \le i, j \le D)$$

$$E^*_{ij} V = \mathrm{Span}\left(\hat{\omega} \mid \omega \in X,\ \partial(x,\omega) = i \ \ \partial(y,\omega) = j \right) \qquad (0 \le i, j \le D).$$

Let δ denote all 1's vector in V.

Let

$$\delta_{ij} = E^*_{ij}\delta = \sum_{\substack{\omega \in X \\ \partial(x,\omega)=i,\ \partial(y,\omega)=j}} \hat{\omega}.$$

Now

$$\Delta \hat{y} \in E^*_1(x) V = \underset{\substack{\rotatebox{90}{$\in$}}}{E^*_{10}(x) V} + \underset{\substack{\rotatebox{90}{$\in$}}}{E^*_{11} V} + \underset{\substack{\rotatebox{90}{$\in$}}}{E^*_{12} V} \quad \text{(orthogonal direct sum)}$$

So $\Delta \hat{y} = \delta^+_{10} + \delta^+_{11} + \delta^+_{12}$

Observe : $\hat{z} \in E^*_{12} V$ is not orthogonal to $\Delta \hat{y}$.

So $\delta^+_{12} \neq 0$ — (2)

Observe : $R^{i-1}(\delta^+_{01} + \delta^+_{11} + \delta^+_{12})$

$$= R^{i-1} \Delta \hat{y}$$

$$= R^{i-1} (R^{-1})^{i-1} E^*_i A_i E^*_1 \hat{y}$$

$$= E^*_i A_i E^*_1 \hat{y}$$

[$\boxed{\text{HS}}$ (because on each irreducible thin module with standard basis $w_r, w_{r+1}, \cdots, w_{r+d}$,

$R^{-1} w_i = w_{i-1} \quad i > r \qquad R^{-1} w_r = 0$, and

$E^*_1 V$ is a orthogonal direct sum of irred : $r \le 1$.)]

$$= \delta_{ii} \in E_{ii} V.$$

But we can control $R^{i-1}\delta_{10}^{+}$, $R^{i-1}\delta_{11}^{+}$ also.

<u>Claim</u> $R E_{jj}^{*} V \subseteq E_{j+1,j+1}^{*} V + E_{j+1,j}^{*} V$. $(1 \leq j \leq D-1)$

<u>Proof of Claim</u> Clear.

By Claim

$$R^{i-1}\delta_{10}^{+} \in E_{i\,i-1}^{*} V \quad \text{and}$$

$$R^{i-1}\delta_{11}^{+} \in E_{i\,i-1}^{*} V + E_{ii}^{*} V.$$

Hence, we conclude that

$$R^{i-1}\delta_{12}^{+} = R^{i-1}\Delta\hat{y} - R^{i-1}\delta_{10}^{+} - R^{i-1}\delta_{11}^{+}$$
$$\in E_{i\,i-1}^{*} V + E_{ii}^{*} V$$

But now

$$\begin{aligned} 0 &= E_{i\,i+1}^{*} R^{i-1}\delta_{12}^{+} \\ &= E_{i\,i+1}^{*} A^{i-1} E_{12}^{*} \delta_{12}^{+} \\ &= R(y)^{i-1}\delta_{12}^{+} \qquad \text{— (3)} \end{aligned}$$

By LEMMA 55 (ii) (Lec 32-1),

$$R(y)^{i-1} : E_2^{*}(y) V \rightarrow E_{i+1}^{*} V$$

is 1-1, since Γ is thin and $i-1 \leq D-4$.

So $\delta_{12}^{+} = 0$ by (3).

But this contradicts (2). Hence our assumption $\Delta_{yz} \neq 0$ is false, and the condition (i) of the definition of generalized adjacency matrices is satisfied. This proves the lemma.

Lecture 38 Mon. May 3, 1993
(originally 65)

LEMMA 68 Let $\Gamma = (X, E)$ be a thin distance regular graph of diameter $D \geq 5$, Q-polynomial w.r.t. $E_0, E_1, \cdots, E_D$.
Pick $x, y \in X$ s.t. $\partial(x, y) = 1$.
Write $E^*_{ij} := E^*_i(x) E^*_j(y)$ $(0 \leq i, j \leq D)$

(i) $E^*_{22} A E^*_{11} : E^*_{11} V \longrightarrow E^*_{22} V$ is 1-1.

(ii) $\forall z \in X$ s.t. $\partial(x, z) = \partial(y, z) = 1$. $\exists w \in X$, s.t.
$\partial(w, x) = \partial(w, y) = 2$. $\partial(w, z) = 1$.

Proof.
(i) Write $E^*_i \equiv E^*_i(x)$, $R \equiv R(x)$, $F \equiv F(x)$ etc.
Suppose
$$\exists\, 0 \neq v \in E^*_{11} V \quad \text{s.t.} \quad E^*_{22} A E^*_{11} v = 0 \qquad \text{---}(1)$$

Claim 1 $E^*_{34} A^2 E^*_{12} A E^*_{11} v \neq 0$

Proof of Claim 1
Recall by LEMMA 55 (ii) $(3 \leq 5-2 \leq D-2t)$
$$R(y)^3 : E^*_1(y) V \to E^*_4(y) V$$
is 1-1.
Since $v \in E^*_1(y) V$, we find
$$\begin{aligned} 0 &\neq R^3(y) v \\ &= E^*_4(y) A^3 E^*_1(y) v \\ &= E^*_4(y) A^2 E^*_2(y) A E^*_{11} v \\ &= E^*_4(y) A^2 \Big(\sum_{h=0}^{D} E^*_{h2} \Big) A E^*_{11} v \\ &= E^*_4(y) A^2 (E^*_{12} + E^*_{22}) A E^*_{11} v \end{aligned}$$

$$= E_4^*(y) A^2 E_{12}^* A E_{11}^* v \qquad \text{by (1)}$$
$$= E_{34}^*(y) A^2 E_{12}^* A E_{11}^* v$$

This proves the claim

By Theorem 51 (i) (Lec 30-3)

$$0 = (g_3^- R^2 F + RFR + g_3^+ FR^2 - \gamma R^2) E_1^* \qquad \text{— (2)}$$

$\boxed{\text{HS}}$ Thm 51 (i)

$$(g_i^- FL^2 + LFL + g_i^+ L^2 F - \gamma L^2) E_i^* = 0 \qquad (2 \le i \le D)$$

$i = 3$

$$E_1^* (g_3^- FL^2 + LFL + g_3^+ L^2 F - \gamma L^2) E_3^* = 0$$

Taking the transpose we have

$$E_3^* (g_3^- R^2 F + RFR + g_3^+ FR^2 - \gamma L^2) E_1^* = 0$$

Hence we have (2).

Multiply each term on left by $E_4^*(y)$, on right by $E_1^*(y)$. we find

$$0 = g_3^- E_{34}^* R^2 F E_{11}^* + E_{34}^* RFR E_{11}^* + g_3^+ E_{34}^* FR^2 E_{11}^*$$
$$- \gamma E_{34}^* R^2 E_{11}^*$$
$$= g_3^- E_{34}^* A^2 E_{12}^* A E_{11}^* + E_{34}^* A E_{23}^* A E_{22}^* A E_{11}^*$$
$$+ g_3^+ E_{34}^* A E_{33}^* A E_{22}^* A E_{11}^* \qquad \text{— (3)}$$

Applying this to v, we find by (1) that

$$0 = g_3^- E_{34}^* A^2 E_{12}^* A E_{11}^* v.$$

So $g_3^- = 0$ by Claim 1. But

$$g_3^- = \frac{\theta_1^* - \theta_0^*}{\theta_1^* - \theta_3^*} \qquad \text{by Lemma 52,}$$

$\neq 0$ a contradiction

Let Γ, x, y be as in LEMMA 68
We saw in Lemma 67 (Lec37-5)

$$R^{-1} E_2^* A_2 E_1^* \hat{y} = \delta_{10}^+ + \delta_{11}^+,$$

where $\delta_{10}^+ \in E_{10}^* V = \mathrm{Span}(\hat{y})$
$\delta_{11}^+ \in E_{11}^* V$

DEF. $\psi = \psi(x,y) \in \mathbb{C}$ by

$$\delta_{10}^+ = \psi \hat{y}.$$

We will show that $\psi(x,y)$ is independent of x, y.

Observe. R^{-1}, A_i, $E_i^* \in \mathrm{Mat}_X(\mathbb{Q})$
So $\psi \in \mathbb{Q}$.

1^{st}, show $\psi(x,y) = \psi(y,x)$:

LEMMA 69 (originally Lemma 66)
With the notation of LEMMA 68

(i) $E_{22}^* A E_{11}^* \delta_{11}^+ = \delta_{22}^-$

(ii) $E_{21}^* A E_{11}^* \delta_{11}^+ = -\psi(x,y) \delta_{21}$

(iii) $\langle \delta_{11}^+, \delta_{11} \rangle = \frac{a_2}{c_2} - \psi(x,y)$

(iv) $\psi(x,y) = \psi(y,x)$

(v) $E_{12}^* A E_{11}^* \delta_{11}^+ = -\psi(x,y) \delta_{12}$

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Proof

Write $\psi \equiv \psi(x,y)$.

$R \equiv R(x)$, $E_i^* \equiv E_i^*(x)$ etc.

(i) $R(\delta_{11}^+ + \psi\hat{y}) = R(\delta_{11}^+ + \delta_{10}^+)$

$= R(R^{-1}(E_2^* A_2 E_1^*))\hat{y}$

$= E_2^* A_2 E_1^* \hat{y}$

$= \delta_{22}$

So $\delta_{22} = R(\delta_{11}^+ + \psi\hat{y})$

$= E_2^* A E_1^* (\delta_{11}^+ + \psi\hat{y})$

$= E_{22}^* A E_{11}^* \delta_{11}^+ + \underbrace{\psi E_{22}^* A E_{10}^* \hat{y}}_{= 0}$

(ii) $0 = E_{21}^* \delta_{22}$

$= E_{21}^* R(\delta_{11}^+ + \psi\hat{y})$

$= E_{21}^* A E_{11}^* \delta_{11}^+ + \psi E_{21}^* A E_{10}^* \hat{y}$

$= E_{21}^* A E_{11}^* \delta_{11}^+ + \psi\delta_{21}$

(iii) $p_{22}' = \| \delta_{22} \|^2$

$= \langle \delta_{22}, \delta_{21} + \delta_{22} + \delta_{23} \rangle$

$= \langle R(\delta_{11}^+ + \psi\hat{y}), \delta_{21} + \delta_{22} + \delta_{23} \rangle$

$= \langle \delta_{11}^+ + \psi\hat{y}, L(\delta_{21} + \delta_{22} + \delta_{23}) \rangle$

$= b_1 \langle \delta_{11}^+ + \psi\hat{y}, \delta_{10} + \delta_{11} + \delta_{12} \rangle$

$= b_1 (\langle \delta_{11}^+, \delta_{11} \rangle + \psi)$

So $\langle \delta_{11}^{+}, \delta_{11} \rangle = b_1^{-1} p_{22}^{1} - \psi$

$$= \frac{a_2}{c_2} - \psi$$

$$\left[\boxed{\text{HS}} \quad b_1^{-1} p_{22}^{1} = b_1^{-1} \frac{k_1}{k_1} p_{22}^{1} = b_1^{-1} \frac{1}{k_1} k_2 p_{12}^{2} = b_1^{-1} \frac{b_1}{c_2} a_2 = \frac{a_2}{c_2} \right]$$

(iv) Interchanging roles of x, y above, we find

$\exists\, \delta_{11}^{+\prime} \in E_{11}^{*}V$ s.t

$$R(y)^{-1} E_2^{*}(y) A_2 E_1^{*}(y) \hat{x} = \delta_{11}^{+\prime} + \psi(y,x)\hat{x}.$$

Then

$$E_{22}^{*} A E_{11}^{*} (\delta_{11}^{+\prime}) = \delta_{22}.$$

So

$$E_{22}^{*} A E_{11}^{*} (\delta_{11}^{+} - \delta_{11}^{+\prime}) = 0$$

Hence $\delta_{11}^{+} = \delta_{11}^{+\prime}$ since

$E_{22}^{*} A E_{11}^{*} : E_{11}^{*}V \to E_{22}^{*}V$ is 1-1.

Now

$$\frac{a_2}{c_2} - \psi(x,y) = \langle \delta_{11}^{+}, \delta_{11} \rangle$$

$$\parallel$$

$$\frac{a_2}{c_2} - \psi(x,y) = \langle \delta_{11}^{+\prime}, \delta_{11} \rangle$$

Thus $\psi(x,y) = \psi(y,x)$.

(v) Immediate from (ii), (iv).

Lecture 39 Wed May 5, 1993.

Assume $\Gamma = (X, E)$ is thin, distance regular of diameter $D \geq 5$,
Q-polynomial w.r.t. $E_0, \dots, E_D$.
Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $R \equiv R(x)$, $T \equiv T(x)$

Pick $y \in X$, $\partial(x,y) = 1$. Write $E_{ij}^* \equiv E_i^*(x) E_j^*(y)$

$$\delta_{ij} = E_{ij}^* \delta \qquad \tilde{A} = E_1^* A E_1^*$$

Recall

$$R^{-1} E_2^* A_2 E_1^* \hat{y} = \underset{\substack{\cap \\ E_{11}^* V}}{\delta_{11}^+} + \psi(x,y)\hat{y}$$

Saw $\psi(x,y) = \psi(y,x)$.
We show below that
$\psi(x,y)$ is independent of edge x, y.

LEMMA 70 (Originally Lemma 67).
With the above notation, set $\psi := \psi(x,y)$.

(i) $\delta_{11}^- := \tilde{A}\delta_{11}^+ - \left(\frac{a_2}{c_2} - \psi\right)\hat{y} + \psi\delta_{12} \in E_{11}^* V$

(ii) $\delta_{11}^-(x,y) = \delta_{11}^-(y,x)$.

Proof

(i) $$\tilde{A}\delta_{11}^+ = \underset{\substack{\cap \\ E_{12}^* V}}{\delta_{12}^-} + \underset{\substack{\cap \\ E_{11}^* V}}{\delta_{11}^-} + \underset{\substack{\cap \\ E_{10}^* V}}{\delta_{10}^-} \qquad \text{--- (1)}$$

$$\delta_{12}^- = E_{12}^* A E_{11}^* \delta_{11}^+$$
$$= -\psi(x,y)\,\delta_{12} \qquad \text{--- (2)}$$

by LEMMA 69 (v)

Also

$$\delta_{10}^{-} = \sigma \hat{y} \qquad \text{some } \sigma \in \mathbb{C}, \quad \text{where}$$

$$\begin{aligned} \sigma &= \langle \tilde{A} \delta_{11}^{+}, \hat{y} \rangle \\ &= \langle \delta_{11}^{+}, \tilde{A} \hat{y} \rangle \\ &= \langle \delta_{11}^{+}, \delta_{11} \rangle \\ &= \frac{a_2}{c_2} - \psi \qquad \text{—— (2)} \end{aligned}$$

Solving for δ_{11}^{-} in (1), using (2), (3), we have

$$\begin{aligned} \delta_{11}^{-} &= \tilde{A} \delta_{11}^{+} - \delta_{12}^{-} - \delta_{10}^{-} \\ &= A \delta_{11}^{+} + \psi \delta_{12} - \left(\frac{a_2}{c_2} - \psi \right) \hat{y}. \end{aligned}$$

(ii) Since

$$\delta_{11}^{-} = E_{11}^{*} A E_{11}^{*} \delta_{11}^{+},$$

we have $\delta_{11}^{+}(x, y) = \delta_{11}^{+}(y, x)$.

LEMMA 71 (originally Lemma 68)

With the above notation

$\psi := \psi(u, v)$ is independent of u, v

($u, v \in X$, $\partial(u, v) = 1$)

Proof.

Let x, y be as above $(x \sim y)$, and pick $z \in X$ s.t. $\partial(x, z) = 1$ but $z \neq y$.

Then it suffices to show

$$\psi(x, y) = \psi(x, z)$$

Case: $\partial(y, z) = 2$:

Set $\Delta := \tilde{A} R^{-1} E_2^* A_2 E_1^*$

Observe: $\Delta \in E_1^* T E_1^*$

and $E_1^* T E_1^*$ is symmetric

by Lemma 61 (Lec 33-7)

Hence $\Delta_{yz} = \Delta_{zy}$.

Since $\Delta \in \mathrm{Mat}_X(\mathbb{R})$,

$$\langle \Delta \hat{y}, \hat{z} \rangle = \langle \Delta \hat{z}, \hat{y} \rangle$$

But

$$\langle \Delta \hat{y}, \hat{z} \rangle = \langle \tilde{A} \delta_{11}^+ + \psi(x,y)\hat{y}, \hat{z} \rangle$$

$$= \langle \tilde{A} \delta_{11}^+, \hat{z} \rangle \quad (\text{as } \partial(x,y)=2)$$

$$= \left\langle \delta_{11}^- + \left(\frac{a_2}{c_2} - \psi\right)\hat{y} - \psi(x,y)\delta_{12}, z \right\rangle \quad (\text{by Lemma 70(i) Lec 39-1})$$

$$= -\psi(x, y)$$

Similarly, $\langle \Delta \hat{z}, \hat{y} \rangle = -\psi(x, z)$.

Hence $\psi(x, y) = \psi(x, z)$.

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Case $\partial(y,z)=1$: By LEMMA 68 (ii),

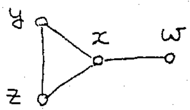
there exists $w \in X$ s.t

$\partial(x,w)=1, \quad \partial(w,y)=2, \quad \partial(w,z)=2.$

Now

$\psi(x,y) = \psi(x,w) = \psi(x,z)$

from the first case.



LEMMA 72 (originally LEMMA 69)

With the above notation,

(i) $A^+ := R^{-1}E_2^*A_2E_1^* - \psi E_1^*$

(ii) $A^- := \tilde{A}A^+ - \left(\frac{a_2}{c_2} - \psi\right)E_1^* + \psi\left(\tilde{J} - \tilde{A} - E_1^*\right)$

are both generalized adjacency matrices for the subgraph induced on the 1^{st} subconstituent with respect to x.

Moreover, A^+, A^- have 0 diagonal.

Proof.

Pick $y, z \in X$ s.t. $\partial(x,y) = \partial(x,z) = 1$.

Show A^+_{yz}, A^-_{yz} are both 0

if $\partial(y,z) = 0$ or 2.

Since $A^+\hat{y} = R^{-1}E_2^*A_2E_1^*\hat{y} - \psi E_1^*\hat{y} = \delta^+_{11}$,

$$A^+_{yz} = \langle A^+\hat{y}, \hat{z}\rangle = \langle \delta^+_{11}, \hat{z}\rangle = 0$$

if $\partial(y,z)=0$ or 2.

Since $A^-\hat{y} = \tilde{A}A^+\hat{y} - \left(\frac{a_2}{c_2} - \psi\right)E_1^*\hat{y} + \psi(\tilde{J} - \tilde{A} - E_1^*)\hat{y}$

$= \tilde{A}\delta_{11}^+ - \left(\frac{a_2}{c_2} - \psi\right)\hat{y} + \psi\delta_{12}$

$= \delta_{11}^-$,

$$A^-_{yz} = \langle A^-\hat{y}, \hat{z}\rangle$$
$$= \langle \delta_{11}^-, \hat{z}\rangle$$
$$= 0$$

if $\partial(y, z) = 0$ or 2.

Since $E_1^*TE_1^* = \text{Span}(\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2, \cdots)$.
by LEMMA 61 (Lec 33-7).
A^+, A^- are both generalized matrices for the subgraph induced on the 1st subconstituent with respect to x.

Summary

$E_1^*TE_1^* \ni \tilde{J}, E_1^*, \tilde{A}, A^+, A^-$

and $\dim E_1^*TE_1^* \leq 5$.

~~Fact:~~ With the above assumptions

$E_1^*TE_1^* = \text{Span}(\tilde{J}, E_1^*, \tilde{A}, A^+, A^-)$

(may not be independent)

LEMMA 73 (originally 70).

$$T(y)\hat{x} = T(x)\hat{y} \qquad (\text{Here } \partial(x,y)=1)$$

Proof

$$
\begin{aligned}
T(x)\hat{y} &= T(x)E_1^*\hat{y} \\
&= M(E_0^* + E_1^*)T(x)\,E_1^*\hat{y} \qquad (\text{as } \Gamma \text{ is thin}) \\
&= M\hat{x} + ME_1^*TE_1^*\hat{y} \\
&= M\hat{x} + M\,\mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, A^+, A^-)\,\hat{y} \\
&= M\hat{x} + M\,\mathrm{Span}(\delta_{12}+\delta_{11}+\delta_{10}, \delta_{10}, \delta_{11}, \delta_{11}^+, \delta_{11}^-) \\
&= M\,\mathrm{Span}(\delta_{01}, \delta_{10}, \delta_{11}, \delta_{11}^+, \delta_{11}^-)
\end{aligned}
$$

But the identity of these conditions does not change if we interchange x, y.

Hence

$$T(x)\hat{y} = T(y)\hat{x}.$$

Lecture 40 Fri. May 7, 1993 (Last Class Day)

LEMMA 74 (originally Lemma 71).

With the above notation

Let W denote thin irreducible T-module of endpoint 0 or 1.

Pick $0 \neq v \in E_1^* V$.

		$r(W)=1$	$r(W)=0$
(i) eigenvalue for	$\tilde{J}$	0	k
(ii)	E_1^*	1	1
(iii)	$\tilde{A}$	$a_0(W)$	a_1
(iv)	A^+	$a^+(W)=\frac{\gamma_1}{c_2}-1-\psi$	$\frac{a_2}{c_2}-\psi$
(v)	A^-	$a^-(W)=a_0(W)\left(\frac{\gamma_1}{c_2}-1-2\psi\right)-\frac{a_2}{c_2}$	ϕ L40-3

where $\gamma_0 = 1 + a_0(W)$

$$\gamma_1 = \frac{c_2 b_2 \gamma_0 -}{b_1 + \gamma_0(a_1+2-c_2) - \gamma_0^2}$$ as in Theorem 32 (Lec 16-4)

Proof

(i) - (iii) clear

(iv) $A^+ = R^{-1}E_2^* A_2 E_1^* - \psi E_1^*$

$$A_2 = \frac{A^2 - a_1 A - kI}{c_2}$$

$$E_2^* A_2 E_1^* = E_2^*\left(\frac{A^2 - a_1 A - kI}{c_2}\right)E_1^*$$

$$= \frac{1}{c_2}(RF + FR - a_1 R)E_1^*$$

$r(W)=1$ $\quad A^+v = \frac{1}{c_2}\{R^{-1}RFv + R^{-1}FRv - R^{-1}a_1Rv\} - \psi v$

$$= \frac{1}{c_2}\{R^{-1}R\,a_0(W)v + R^{-1}a_1(W)Rv - a_1R^{-1}Rv\} - \psi v$$

$$= v\left\{\frac{1}{c_2}(a_0(W) + a_1(W) - a_1) - \psi\right\}$$

But $a_1(W) = \gamma_1 = \gamma_0 + a_1 + 1 - c_2$, $\gamma_0 = a_0(W)+1$
by Theorem 32 (Lec 16-4)

So

$$A^+v = v\left\{\frac{1}{c_2}(a_0\cancel{(W)} + \gamma_1 - \cancel{\gamma_0} + \cancel{a_1} \cancel{+1} - c_2 - \cancel{a_1}) - \psi\right\}$$

$$= v\left\{\frac{\gamma_1}{c_2} - 1 - \psi\right\}$$

$r(W)=0$ $\quad A^+v = \frac{1}{c_2}\{R^{-1}RFv + R^{-1}FRv - a_1R^{-1}Rv\} - \psi v$

$$= \frac{1}{c_2}\{R^{-1}R\overset{a_1(W)}{\overset{\shortparallel}{a_1}}v + R^{-1}\overset{a_2(W)}{\overset{\shortparallel}{a_2}}Rv - a_1R^{-1}Rv\} - \psi v$$

$$= \left(\frac{a_2}{c_2} - \psi\right)v$$

(v) Immediate from (iv) and

$$A^- = \tilde{A}A^+ - \left(\frac{a_2}{c_2} - \psi\right)E_1^* + \psi(\tilde{J} - \tilde{A} - E_1^*)$$

$\boxed{\text{HS}}$ $\quad r(W)=1$

$$A^-v = \left(a_0(W)\left(\frac{\gamma_1}{c_2} - 1 - \psi\right) - \left(\frac{c_2}{a_2} - \psi\right) + \psi(-a_0(W) - 1)\right)v$$

$$= \left(a_0(W)\left(\frac{\gamma_1}{c_2} - 1 - 2\psi\right) - \frac{c_2}{a_2}\right)v$$

$\boxed{\text{HS}}$ $r(W)=0$

$$A^{-}v = \Big(a_1\Big(\frac{a_2}{c_2}-\psi\Big)-\Big(\frac{a_2}{c_2}-\psi\Big)+\psi(\underbrace{k-a_1-1}_{})\Big)v$$

$$\overset{b_1}{\underset{\|}{}}$$

$$= \Big((a_1-1)\frac{a_2}{c_2} + (k-2a_1)\psi\Big)v$$

Let W_1, W_2, W_3, W_4 denote 4 possible isomorphism classes of T-modules of endpoint 1. $a_0(W_1), \cdots, a_0(W_4)$ are roots of 4th degree polynomial whose coefficients are determined form intersection numbers of Γ. ?

So $a_0(W_1), \cdots, a_0(W_4)$ are determined by intersection numbers.

Let $\tilde{m}_i$ denote the multiplicity of W_i $(1 \le i \le 4)$
($=$ multiplicity of $a_0(W)$ as eigenvalue of $\tilde{A}|_{(E_1^*V)_{new}}$)

LEMMA 75 (originally Lemma 72)
With the above notation

(i) $\tilde{m}_1, \cdots, \tilde{m}_4$ are determined from intersection numbers and ψ.

(ii) $\tilde{m}_i$ is independent of vertex x. $(1 \le i \le 4)$

(iii) $l := \dim E_1^* T E_1^*$ is independent of x.

Proof

(i) Let $e_i \in E_1^* T E_1^*$ $(1 \le i \le 4)$ denote the orthogonal projection onto the maximal eigenspace of $(E_1^* V)_{new}$ corresponding to λ_i
($e = 0 \Leftrightarrow \lambda_i$ does not appear)

Set $e_0 = \frac{1}{k} \tilde{J}$.

	e_0	e_1	e_2	e_3	e_4
$\tilde{J}$	k	0	0	0	0
E_1^*	1	1	1	1	1
$\tilde{A}$	a_1	$a_0(W_1)$	$a_0(W_2)$	$a_0(W_3)$	$a_0(W_4)$
A^+	$\left(\frac{a_2}{c_2} - \psi\right)$	$a^+(W_1)$	$a^+(W_2)$	$a^+(W_3)$	$a^+(W_4)$
A^-	$(\quad)$	$a^-(W_1)$	$a^-(W_2)$	$a^-(W_3)$	$a^-(W_4)$

Observe $e_i^2 = e_i$. trace e_i = rank $e_i = \tilde{m}_i$ $(1 \le i \le 4)$
trace e_0 = rank $e_0 = 1$.

Take trace of $\tilde{J}$, E_1^*, $\tilde{A}$, A^+, A^-.
We have

$$k = k$$
$$k = 1 + \tilde{m}_1 + \tilde{m}_2 + \tilde{m}_3 + \tilde{m}_4$$
$$0 = a_1 + a_0(W_1)\tilde{m}_1 + a_0(W_2)\tilde{m}_2 + a_0(W_3)\tilde{m}_3 + a_0(W_4)\tilde{m}_4$$
$$0 = \left(\frac{a_2}{c_2} - \psi\right) + a^+(W_1)\tilde{m}_1 + a^+(W_2)\tilde{m}_2 + a^+(W_3)\tilde{m}_3 + a^+(W_4)\tilde{m}_4$$
$$0 = (\quad) + a^-(W_1)\tilde{m}_1 + a^-(W_2)\tilde{m}_2 + a^-(W_3)\tilde{m}_3 + a^-(W_4)\tilde{m}_4$$

The coefficient matrix for $\tilde{m}_1, \dots, \tilde{m}_4$ is nonsingular.
(this is what you need to check to show.)

$\boxed{HS}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ a_0(W_1) & a_0(W_2) & a_0(W_3) & a_0(W_4) \\ a^+(W_1) & a^+(W_2) & a^+(W_3) & a^+(W_4) \\ a^-(W_1) & a^-(W_2) & a^-(W_3) & a^-(W_4) \end{bmatrix} = \begin{bmatrix} 1 \\ a_0(W_i) \\ \frac{\gamma_1(W_i)}{c_2} - 1 - \psi \\ a_0(W_i)\left(\frac{\gamma_1}{c_2} - 1 - 2\psi\right) - \frac{a_2}{c_2} \end{bmatrix}$$

$$\doteq \begin{bmatrix} 1 \\ a_0(W_i) \\ \frac{\gamma_1(W_i)}{c_2} - 1 - \psi \\ a_0(W_i)\left(\frac{\gamma_1(W_i)}{c_2} - 1 - 2\psi\right) - \frac{a_2}{c_2} \end{bmatrix} \sim \begin{bmatrix} 1 \\ a_0(W_i) \\ \gamma_1(W_i) \\ a_0(W_i)\gamma_1(W_i) \end{bmatrix}$$

$$\gamma_0(W_i) = a_0(W_i) + 1$$

$$\gamma_1(W_i) = \frac{c_2 b_2 \gamma_0(W_i)}{x_1(W_i)} \qquad x_1(W_i) = b_1 + \gamma_0(W_i)(a_1 + 1 - c_2 - a_0(W_i))$$

Dividing by $x_1(W_1)x_1(W_2)x_1(W_3)x_1(W_4)(c_2 b_2)^2$ we have

$$\doteq \begin{bmatrix} x_1(W_i) \\ a_0(W_i)x_1(W_i) \\ \gamma_0(W_i) \\ a_0(W_i)\gamma_0(W_i) \end{bmatrix} = \begin{bmatrix} b_1 + (a_1+2-c_2)\gamma_0(W_i) - \gamma_0(W_i)^2 \\ (-1+\gamma_0(W_i))\left(b_1 + (a_1+2-c_2)\gamma_0(W_i) - \gamma_0(W_i)^2\right) \\ \gamma_0(W_i) \\ (-1+\gamma_0(W_i))\gamma_0(W_i) \end{bmatrix}$$

$$= b_1 \begin{bmatrix} 1 \\ -\gamma_0(W_i)^3 \\ \gamma_0(W_i) \\ \gamma_0(W_i)^2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ \gamma_0(W_i) \\ \gamma_0(W_i)^2 \\ \gamma_0(W_i)^3 \end{bmatrix} \neq 0.$$

(ii) ψ is independent of base vertex x.

(iii) $\dim E_1^* T E_1^* = |\{ i \mid 1 \le i \le 4,\ e_i \neq 0 \}| + 1$

$= |\{ i \mid 1 \le i \le 4,\ \tilde{m}_i \neq 0 \}| + 1$

Let $\Gamma = (X, E)$ be thin distance regular of diameter $D \geq 5$.
θ-polynomial w.r.t. $E_0, E_1, \cdots, E_D$.

Fix $x, y \in X$, $\partial(x, y) = 1$

$$E^*_{ij} \equiv E^*_i(x) E^*_j(y) \qquad \delta_{ij} = E^*_{ij} \delta$$

We saw $T(x)\hat{y} = T(y)\hat{x}$.

Hence

$$H := T(x)\hat{y} = T(y)\hat{x}$$

is $T(x, y)$-module

$T(x, y) \subseteq Mat_X(\mathbb{C})$ is generated by

$$M, M^*(x), M^*(y).$$

LEMMA 76 (originally Lemma 73).

With the above notation

(i) $E^*_{i\,i+1} H = \mathrm{Span}(\delta_{i, i+1})$ $\quad (0 \leq i \leq D-1)$

(ii) $E^*_{i+1, i} H = \mathrm{Span}(\delta_{i+1, i})$ $\quad (0 \leq i \leq D-1)$

(iii) $\dim E^*_{ii} H = \ell - 2 \leq 3$ $\quad (1 \leq i \leq D-1)$

Proof.

(i) $\supseteq$: $\delta_{i\,i+1} = E^*_i A_{i+1} \hat{y}$

$$\in T(x) y = H$$

$\subseteq$: Pick $h \in E^*_{i\,i+1} H$.

$$h = R^{i-1} v, \text{ where } v = (R^{-1})^{i-1} h \in E^*_1 V$$

So $\quad v \in \mathrm{Span}(\delta_{12}, \delta_{11}, \delta_{10}, \delta_{11}^{+}, \delta_{11}^{-})$

[$\boxed{\text{HS}}$ $\quad v \in E_1^* V \cap T(x)\hat{y} = E_1^* T(x) E_1^* \hat{y}$

$= \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, A^+, A^-)\hat{y}$

$= \mathrm{Span}(\delta_{10}+\delta_{11}+\delta_{12}, \delta_{10}, \delta_{11}, \delta_{11}^{+}, \delta_{11}^{-})$

$= \mathrm{Span}(\delta_{10}, \delta_{11}, \delta_{12}, \delta_{11}^{+}, \delta_{11}^{-})$]

Hence $\quad \exists \alpha \in \mathbb{C} \quad$ s.t.

$v - \alpha\delta_{12} \in \mathrm{Span}(\delta_{11}, \delta_{10}, \delta_{11}^{+}, \delta_{11}^{-})$

$= E_{11}^* H + E_{10}^* H$

So $\quad v - \alpha(\delta_{12}+\delta_{11}+\delta_{10}) \in E_{11}^* H + E_{10}^* H.$

$$R^{i-1}(v - \alpha(\delta_{12}+\delta_{11}+\delta_{10})) \in E_{ii}^* H + E_{i\,i-1}^* H$$
$\|$
$h - \alpha'(\delta_{i\,i+1} + \delta_{ii} + \delta_{i\,i-1})$

Hence

$h - \alpha'\delta_{i\,i+1} \in (E_{ii}^* H + E_{i\,i-1}^* H) \cap E_{i\,i+1}^* H$

Thus $\qquad h = \alpha'\delta_{i\,i+1} \in \mathrm{Span}(\delta_{i\,i+1})$

(ii) By symmetry, we have the assertion.

(iii) $\quad E_i^* H = E_{i\,i+1}^* H + E_{ii}^* H + E_{i\,i-1}^* H$

dim $\quad l \qquad\qquad 1 \qquad\qquad l-2 \qquad\qquad 1$

[$\boxed{\text{HS}}$ $\quad$ Since $\quad H = T(x)\hat{y} \subseteq T(x) E_i^*(x) V.$

$(R^{-1})^{i-1} : E_i^* H \to E_1^* H \quad$ is 1-1. onto if $i \le D-1$]

THEOREM 77 (originally Theorem 74)

Let $\Gamma = (X, E)$ be thin distance regular of diameter $D \geq 5$.

Q-polynomial w.r.t. $E_0, E_1, \cdots, E_D$.

Pick i $(2 \leq i \leq D)$ and pick $x, y, z \in X$

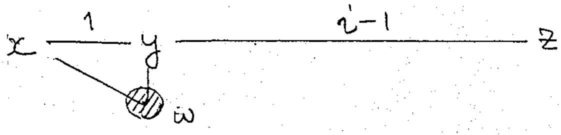
s.t. $\partial(x, y) = 1$, $\partial(y, z) = i-1$ $\partial(x, z) = i$.

Then

$$z_i = \left| \{ w \mid w \in X, \partial(x, w) = 1, \partial(y, w) = 1, \partial(z, w) = i-1 \} \right|$$

is independent of x, y, z.

Proof.



z_i is the zx entry in

$$\Delta = E^*_{i-1}(y) A_{i-1} E^*_1(y) \, A \, E^*_1(y)$$

$$\Delta \hat{x} = \sum_{z \in X \quad \partial(x, z) = i \quad \partial(y, z) = i-1} \cdots \, z_i(x, y, z) \hat{z}.$$

But $\Delta \hat{x} \in E^*_{i-1, i} T(y) x = \mathrm{Span}(\delta_{i-1, i})$

Hence

$z_i(x, y, z)$ is independent of z.

So $z_i(x, y, z)$ is determined by intersection numbers and $\psi = \psi(x, y)$, which is independent of x, y also.

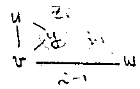
SOME OPEN PROBLEMS CONCERNING DISTANCE-REGULAR GRAPHS, THE THIN CONDITION, AND THE Q-POLYNOMIAL PROPERTY

The questions below are unsolved as of May, 1993 (to my knowledge). A complete solution (or even a significant partial solution in some cases) to any one of these problems would be publishable. I have tried to estimate the level of difficulty of each problem listed below. A $*$ means I believe the problem is relatively easy in the sense that it can be solved using ideas from the course. There are no conceptual gaps to overcome that I am aware of (but the calculations might be quite difficult, however!). A $****$ means I have no idea how to begin to attack the problem. I am only mentioning problems of this kind to give you an idea about what is known in this field.

In what follows, $\Gamma = (X, E)$ denotes a distance-regular graph with diameter $D \geq 3$. Pick $x \in X$, and write $E_i^* = E_i^*(x)$, $T = T(x)$, $R = R(x)$, $F = F(x)$, $L = L(x)$, and consider the following additional properties:

- Dist: Γ is distance-transitive.
- Q: Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents.
- Bip: Γ is bipartite.
- TH: Γ is thin (over the field of complex numbers).
- Few1: The subgraph induced on the first subconstituent of Γ with respect to x has at most 5 distinct eigenvalues.
- Few2: The subgraph induced on the second subconstituent of Γ with respect to x has at most 16 distinct eigenvalues.
- Z: For all integers $i\ (2 \leq i \leq D)$, and all triples u, v, w $(u, v, w \in X)$ such that $\partial(u,v) = 1$, $\partial(v,w) = i-1$, and $\partial(u,w) = i$, the number

$$z_i := |\{\, y \mid y \in X,\ \partial(y,u) = \partial(y,v) = 1,\ \partial(y,w) = i-1 \,\}|$$



is a constant that does not depend on u, v, w.

The following implications are known: $Q + Bip \rightarrow TH$, $Q + TH \rightarrow Few1, Few2, Z$

(1) **** Classify all the distance-regular graphs (with sufficiently large diameter). If neccessary, assume some combination of the above properties. (My personal goal is to classify all the graphs Γ satisfying both Q, TH. I expect this will take a number of years).

(2)** Assume Q, Bip, and classify Γ.

(3)* Find generalizations to the theorems of the course for non-regular, bi-distance-regular, bipartite graphs. DBRG

(4)* Assume Q, and let W denote an irreducible T-module with endpoint 1 that is not thin. Find a nice basis for W and find the matrices representing the adjacency matrix A and the dual adjacency matrix A^* with respect to this basis. Perhaps assume classical parameters. Theorems 51, 54 should be useful.

(5)* Is it true that Γ is thin over the field of complex numbers iff Γ is thin over the field of real numbers? What does it mean for Γ to be thin over the field of rational numbers? The examples suggest that if Γ is thin over the complex numbers then it is already thin over the rational numbers. If this is true, it would be nice to have a proof. For the moment, suppose it is not true. Assume Γ is thin over the field of complex numbers, and define the *splitting field* of Γ to be the minimal extention of the rational field over which Γ is thin. Then the elements of the Galois group of the splitting field act on the standard module, and permute the isomorphism classes of irreducible T-modules. How are the isomorphism classes of T-modules involved related? Can the permuations be nontrivial?

(6)** Assume Q, and assume there is a second Q-polynomial ordering of the primitive idempotents. Prove TH. I believe in this case the first subconstituent has at most 4 distinct eigenvalues, and the constant ψ from class is determined by the intersection numbers. It may be possible to classify all such Γ.

(7)** Assume Q, and assume there is a second P-polynomial ordering of the distance matrices. I believe the same thing happens as in (6) above.

(8)** A path $y = y_0, y_1, \ldots, y_t = z$ in Γ is said to be *geodetic* whenever $\partial(y,z) = t$. Let us say a subset Δ of X is *geodetically closed* whenever all vertices on all geodetic paths with endpoints in Δ are also in Δ. For any vertices $y, z \in X$, observe there exists a unique minimal geodetically closed subset containing y, z, denoted $[yz]$. If the diameter of $[yz]$ equals $\partial(y,z)$, we say $[yz]$ is a *subspace* of Γ. Assume Q, TH, and pick any $y, z \in X$. Show $[yz]$ is a subspace. Furthermore, show the subgraph induced on $[yz]$ is distance-regular, and satisfies Q, TH. If this proves not to be the case, find a simple additional assumption on Γ under which it is true. (It seems to hold for the known examples). I believe these subspaces are the key to an eventual classification of the graphs satisfying Q, TH (and possibly all distance-regular graphs with sufficiently large diameter). In the examples, the partially ordered set of all subspaces, ordered by reverse inclusion, is some classical geometry. There are many classification theorems in the area of finite projective geometry. My hope is that given any Γ, the partially ordered set of all subspaces is some highly regular geometry that can be classified using one of these theorems, leading us to a classification of the original Γ. (By the way, I intend to explore this area in the course I am teaching next fall on partially ordered sets).

(9)** Assume Q, TH. Find a nice basis for $E_2^*TE_2^*$ in a way that generalizes what we did in class for $E_1^*TE_1^*$.

(10)* Assume B, TH, and that the dimension of $E_2^* T E_2^*$ is at most 4. Show that Q holds. Find a nice basis for $E_2^* T E_2^*$.

(11) It is not hard to show that in general

$$c_i \geq c_{i-1} \qquad (1 \leq i \leq D)$$
$$b_i \leq b_{i-1} \qquad (0 \leq i \leq D-1)$$

It is known that if Γ has at least one cycle $y1, y2, y3, y4, y1$ such that $\partial(y1, y3) = \partial(y2, y4) = 2$,then

$$c_i - c_{i-1} + b_{i-1} - b_i \geq a_1 + 2. \qquad (1 \leq i \leq D).$$

This bound has proved to be quite fundamental. For example, the graphs Γ where equality holds for all i all satisfy Q, and in fact they are precisely the graphs of type IIA or IIC (referring to p10,11 in the thick paper I handed out in class). These graphs have all been classified. I have some papers describing some more general bounds of the above sort, but they are unsatisfactory in the sense that the class of graphs for which equality is attained is not interesting, and may even be empty. Hence, one problem(**) is to find a bound that controls the growth of the c_i's and the decrease of the b_i's, where equality is attained for some nice, large class of graphs. Ideally, this class would contain all the known examples of Γ with sufficiently large diameter. or perhaps all the graphs Γ satisfying Q+TH. Specific problem(*): Assume Z and redo the arguments in the above-mentioned papers. Dramatic improvements in the bounds obtained are expected (I did not realise the significance of Z when I wrote the papers). Since Q+TH $\rightarrow$ Z, the new bounds are expected to give important feasibility conditions on the interection numbers of any Γ satisfying Q and TH.

(12)* Explore the class of graphs that are Q-polynomial with respect to each vertex, but not assumed to be distance-regular. Are these graphs in fact distance-regular or bi-distance-regular? (This result would be very esthetically pleasing to me, since as we have seen, the sibling property of being thin does imply distance-regularity or bi-distance-regularity). If the answer to the above question is "no", just what sort of regularity do these graphs have? For a graph that is Q-polynomial with respect to each vertex, how must the orderings of the primitive idempotents associated with adjacent vertices be related? Is it possible for a distance-regular graph to be Q-polynomial with respect to each vertex, but still not be Q-polynomial? (This is a completely new area. Up untill now, the Q-polynomial property was only defined for distance-regular graphs).

(13)** To what extent do the polynomial relations on R, L, F given in Theorem 51 actually characterize the Q-polynomial property? For example, suppose

(i) $L^2FE_i^*$, $LFLE_i^*$, $FL^2E_i^*$, $L^2E_i^*$ are linearily dependent for all i $(2 \le i \le D)$,
(ii) $FLRE_i^*$, $FRLE_i^*$ are linearily dependent for all i $(0 \le i \le D)$, and
(iii) $RL^2E_i^*$, $LRLE_i^*$, $L^2RE_i^*$, $LF^2E_i^*$, $FLFE_i^*$, $F^2LE_i^*$, LFE_i^*, FLE_i^*, LE_i^* are linearily dependent, for all i $(1 \le i \le D)$.

Then does Q hold? what if we assume TH? If not, what other graphs can one get? are they "almost" Q-polynomial in some sense (perhaps many Krein parameters vanish, but not quite enough to imply Q). What is the essential assumption about the coefficients in the above dependencies that is needed to insure Q?

(14)*** Assume Q and TH. Find the abstract structure of the Norton algebra N. My intuition says that this structure can be computed in terms of the interection numbers and a small list of additional parameters such as ψ. The examples suggest that N is "almost associative" in some sense. Specific problem (*) Find the precise structure of the Norton algebra for the examples $J(d,n)$, $J_q(d,n)$,... and find some pattern. The dual of Theorem 51 is relevant to this problem. My intuition says that the idempotents of N should correspond to the subspaces of Γ referred to in problem 8, and that somehow the multiplication operation in N should be related to the meet and join operations in the geometry of subspaces referred to in that problem.

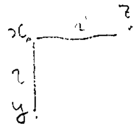
(15)** Assume Q and TH, and pick $y \in X$. show

$$T(x)y = T(y)x.$$

(I can show this for $\partial(x,y) = 1$.) If the above line holds, then apparently $H := T(x)y = T(y)x$ is a module for the algebra $T(x,y)$ generated by the Bose-Mesner algebra M, the dual Bose-Mesner algebra $M^*(x)$, and $M^*(y)$. Observe the elements of $M^*(x)$, $M^*(y)$ mutually commute, and in fact the maximal common eigenspaces of $M^*(x)$, $M^*(y)$ are the E_{ij}^*V $(0 \le i,j \le D)$, where $E_{ij}^* = E_i^*(x)E_j^*(y)$. Find a nice orthogonal basis for each E_{ij}^*H. Observe the union B of these bases is a basis for H. Find the matrices representing A, $A^*(x)$ $A^*(y)$ with respect to B. Choose B so that the entries in these matrices are nice, factorable expressions in the intersection numbers and whatever other parameters are needed. In the case $\partial(x,y) = 1$, these entries can be determined from the intersection numbers and the parameter ψ. If $\partial(x,y) \ge 2$, presumably there are some more free parameters analogous to ψ that play a role. My intuition says that as a $T(x,y)$-module, H is determined from the intersection numbers of Γ and t free parameters, where $t = \partial(x,y)$.

(16)** Does TH and Few1 imply Z? If not, what extra assumption is needed?

(17)** Does TH, Few1, Few2, imply Q? If not, what extra assumption is needed?



(18)** Let Γ be an arbitrary graph, not assumed to be distance-regular. Conjecture: Γ is thin iff for all integers i, j, k, and all vertices x, y, $z \in X$ such that $\partial(x,y) = \partial(x,z) = i$, the number of vertices $w \in X$ with $\partial(w,x) = j$, $\partial(w,y) = 1$, $\partial(w,z) = k$ equals the number of vertices $w' \in X$ such that $\partial(w',x) = j$, $\partial(w',z) = 1$, $\partial(w',y) = k$. If Γ is assumed to be distance-regular, then the conjecture is true and there is a long proof in the thick paper I handed out in class (Theorem 5.1(iii)). A short, slick proof (assuming distance-regularity or not) is very much needed. If the conjecture turns out not to be true in the bi-distance-regular case, find some similar combinatorial characterization of the thin property.

There are a number of additional problems in section 7 of the thick paper I handed out in class. Essentially all the known examples of thin, Q-polynomial distance-regular graphs are listed in section 6 of that paper.

For each of the above problems, I have a good deal of background information to communicate, but unfortunately in most cases it is not in published form! If you tell me what problem you want to focus on, I can tailor a series of lectures this summer towards communicating what I know on the subject. But one key point: Often "I don't know what I know". If you are constantly asking probing questions of me it makes my job a lot easier: it often reminds me of information that is relevant that I had forgotten, or that I had forgotten was relevant.