Algebra I: Final 2006

June 21, 2006

Division: **ID#:** **Name:**

1. Let H be a nonempty subset of a group G satisfying the following.

$$x^{-1}y \in H \text{ for all } x, y \in H.$$

   (a) Show that $1 \in H$, $x^{-1} \in H$ and $xy \in H$ for all $x, y \in H$. (This proves that $H \leq G$.)

   (b) Show that $HH = H = H^{-1}$.

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2. Let G and G_1 be groups and S a subset of G. Let $\alpha : G \to G_1$ be a homomorphism, and $N = \text{Ker}(\alpha)$.

   (a) For $x, y \in G$, show that $xN = yN \Leftrightarrow \alpha(x) = \alpha(y)$.

   (b) Show that $\alpha^{-1}(\alpha(S)) = SN$.

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3. Let $(\boldsymbol{Z}_{15}^*, \cdot)$ be a multiplicative group; here $\boldsymbol{Z}_{15}^*$ is the set of invertible congruence classes $[a]$ modulo 15, i.e., such that $\gcd\{a, 15\} = 1$.

   (a) Show that $[a]^4 = [1]$ for all $[a] \in \boldsymbol{Z}_{15}^*$. Show also that $a^4 \equiv 1 \pmod{15}$ for all integers a such that $\gcd\{a, 15\} = 1$.

   (b) Using the fact proved in (a), determine whether or not $\boldsymbol{Z}_{15}^*$ is a cyclic group.

   (c) Find all subgroups N in $\boldsymbol{Z}_{15}^*$ of order 4.

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4. Let S_6 be the symmetric group of degree 6, and H a subgroup of S_6 containing an element $\pi = (1, 2, 3, 4, 5, 6)$ of order 6.

   (a) Let $\text{sign} : S_6 \to \{\pm 1\}$ $(\sigma \mapsto \text{sign}(\sigma))$ be the sign function on S_6. Show that $\text{sign}(\pi) = -1$.

   (b) Let $N = \text{Ker}(\text{sign})$. Then show that $|H : H \cap N| = 2$.

   (c) Suppose H is a normal subgroup of S_6. Then show that $|S_6 : H| \leq 3$.

Please write your message: (1) Comments on group theory. Suggestions for improvements of this course.

(2) Comments on the educaion at ICU. Suggestions for improvements.

Algebra I: Final 2006 Solutions

June 21, 2006

1. Let H be a nonempty subset of a group G satisfying the following.

$$x^{-1}y \in H \text{ for all } x, y \in H.$$

   (a) Show that $1 \in H$, $x^{-1} \in H$ and $xy \in H$ for all $x, y \in H$. (This proves that $H \leq G$.)
   Solution. Since $H \neq \emptyset$, we can take an element $x \in H$. By the condition, $1 = x^{-1}x \in H$. Let x be an arbitrary element of H. Since $x, 1 \in H$, $x^{-1} = x^{-1}1 \in H$. For $x, y \in H$, since $x^{-1} \in H$, $xy = (x^{-1})^{-1}y \in H$. Therefore $1 \in H$, $x^{-1} \in H$ and $xy \in H$ for all $x, y \in H$. ■

   (b) Show that $HH = H = H^{-1}$.
   Solution. Since H is a subgroup of G, $HH \subseteq H$ and $H^{-1} \subseteq H$. Since $1 \in H$, $H = 1H \subseteq HH$. Hence $HH = H$. Since $H = (H^{-1})^{-1}$, $H = (H^{-1})^{-1} \subseteq H^{-1}$. Hence $H = H^{-1}$ and $HH = H = H^{-1}$. ■

2. Let G and G_1 be groups and S a subset of G. Let $\alpha : G \to G_1$ be a homomorphism, and $N = \mathrm{Ker}(\alpha)$.

   (a) For $x, y \in G$, show that $xN = yN \Leftrightarrow \alpha(x) = \alpha(y)$.
   Solution. Suppose $xN = yN$. Since $x = x1 \in xN = yN$, there exists $n \in N$ such that $x = yn$. Since $N = \mathrm{Ker}(\alpha)$, $\alpha(x) = \alpha(yn) = \alpha(y)\alpha(n) = \alpha(y)$.
   Next assume that $\alpha(x) = \alpha(y)$. Then $1 = \alpha(x)^{-1}\alpha(y) = \alpha(x^{-1}y)$. Hence $x^{-1}y \in N$. Now using 1 (b),

$$yN = xx^{-1}yN \subseteq xNN = xN = yy^{-1}xN = y(x^{-1}y)^{-1}N \subseteq yN^{-1}N = yNN = yN.$$

   Therefore we have $xN = yN$. ■

   (b) Show that $\alpha^{-1}(\alpha(S)) = SN$.
   Solution. Let $s \in S$ and $n \in N$. Then $\alpha(sn) = \alpha(s)\alpha(n) = \alpha(s) \in \alpha(S)$. Hence $SN \subseteq \alpha^{-1}(\alpha(S))$. Let $x \in \alpha^{-1}(\alpha(S))$. Then $\alpha(x) \in \alpha(S)$. Hence there exists $s \in S$ such that $\alpha(x) = \alpha(s)$. Now by (a), $xN = sN$. Hence $x = x1 \in xN = sN \subseteq SN$. Thus $\alpha^{-1}(\alpha(S)) \subseteq SN$ and $\alpha^{-1}(\alpha(S)) = SN$. ■

3. Let $(\boldsymbol{Z}_{15}^*, \cdot)$ be a multiplicative group; here $\boldsymbol{Z}_{15}^*$ is the set of invertible congruence classes $[a]$ modulo 15, i.e., such that $\gcd\{a, 15\} = 1$.

   (a) Show that $[a]^4 = [1]$ for all $[a] \in \boldsymbol{Z}_{15}^*$. Show also that $a^4 \equiv 1 \pmod{15}$ for all integers a such that $\gcd\{a, 15\} = 1$.
   Solution. Since $\boldsymbol{Z}_{15}^* = \{[1], [2], [4], [7], [-7], [-4], [-2], [-1]\}$, $[2]^2 = [-2]^2 = [7]^2 = [-7]^2 = [4]$ and $[4]^2 = [1]$. Hence $[a^4] = [a]^4 = [1]$ for all $[a] \in \boldsymbol{Z}_{15}^*$. This implies that $a^4 - 1$ is divisible by 15 for all integers a such that $\gcd\{a, 15\} = 1$. ■

   (b) Using the fact proved in (a), determine whether or not $\boldsymbol{Z}_{15}^*$ is a cyclic group.
   Solution. If $\boldsymbol{Z}_{15}^*$ is a cyclic group, there is an element of order $|\boldsymbol{Z}_{15}^*| = 8$. But by (a) every element of $\boldsymbol{Z}_{15}^*$ is of order at most 4. Hence $\boldsymbol{Z}_{15}^*$ is not a cyclic group. ■

(c) Find all subgroups N in $\boldsymbol{Z}_{15}^*$ of order 4.

Solution. Every element of order 4 generates a cyclic subgroup of order 4. So $\langle[2]\rangle = \{[1],[2],[4],[-7]\} = \langle[-7]\rangle$, and $\langle[-2]\rangle = \{[1],[-2],[4],[7]\} = \langle[7]\rangle$ are such subgroups. Suppose N is not cyclic. Then every non-identity element of N is of order 2 as its order must divide the order of N. There are 3 elements of order 2 and those together with the identity element form a subgroup $\{[1],[4],[-4],[-1]\}$ of order 4. Hence these three subgroups are those of order 4 in $\boldsymbol{Z}_{15}^*$. ■

4. Let S_6 be the symmetric group of degree 6, and H a subgroup of S_6 containing an element $\pi = (1,2,3,4,5,6)$ of order 6.

(a) Let sign : $S_6 \to \{\pm 1\}$ $(\sigma \mapsto \text{sign}(\sigma))$ be the sign function on S_6. Show that $\text{sign}(\pi) = -1$.

Solution. Since $\pi = (1,2,3,4,5,6) = (1,6)(1,5)(1,4)(1,3)(1,2)$, $\text{sign}(\pi) = -1$. Note that $\ell(\pi) = 5$. ■

(b) Let $N = \text{Ker}(\text{sign})$. Then show that $|H : H \cap N| = 2$.

Solution. The mapping $\text{sign}_{|H} : H \to \{\pm 1\}$ $(\sigma \mapsto \text{sign}(\sigma))$ is a group homomorphism. Since $id, \pi \in H$, $\text{sign}_{|H}$ is a surjective homomorphism. Hence by the isomorphism theorem, $H/K \simeq \{\pm 1\}$ where $K = \text{Ker}(\text{sign}_{|H}) = H \cap N$. Hence $|H : H \cap N| = |\{\pm 1\}| = 2$. ■

(c) Suppose H is a normal subgroup of S_6. Then show that $|S_6 : H| \leq 3$.

Solution. Let $\sigma \in S_n$. Since H is a normal subgroup of S_n and $\pi \in H$, $H \ni \sigma\pi\sigma^{-1} = \sigma(1,2,3,4,5,6)\sigma^{-1} = (\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),\sigma(6))$. Since σ is arbitrary, H contains all 6 cycles. Moreover $\pi^3 = (1,3)(2,4)(3,6) \in H \setminus N$. Since there are 5! 6-cycles and all 6-cycles are in $\pi(H \cap N)$, $|H \cap N| > 5!$. Thus $|H| > 2 \cdot 5!$. Since $|H|$ must divide $|S_6|$, $|S_6 : H| \leq 3$. ■

We can show in this case that $S_6 = H$.