Algebra I: Final 2012

June 22, 2012

ID#: **Name:**

Quote the following when necessary.

A. Subgroup H of a group G:

$$H \leq G \Leftrightarrow \emptyset \neq H \subseteq G,\ xy \in H \text{ and } x^{-1} \in H \text{ for all } x, y \in H.$$

B. Order of an Element: Let g be an element of a group G. Then $\langle g \rangle = \{g^n \mid n \in \boldsymbol{Z}\}$ is a subgroup of G. If there is a positive integer m such that $g^m = e$, where e is the identity element of G, $|g| = \min\{m \mid g^m = e,\ m \in \boldsymbol{N}\}$ and $|g| = |\langle g \rangle|$. Moreover, for any integer n, $|g|$ divides n if and only if $g^n = e$.

C. Lagrange's Theorem: If H is a subgroup of a finite group G, then $|G| = |G : H||H|$.

D. Normal Subgroup: A subgroup H of a group G is normal if $gHg^{-1} = H$ for all $g \in G$. If H is a normal subgroup of G, then G/H becomes a group with respect to the binary operation $(gH)(g'H) = gg'H$.

E. Direct Product: If $\gcd\{m, n\} = 1$, then $\boldsymbol{Z}_{mn} \approx \boldsymbol{Z}_m \oplus \boldsymbol{Z}_n$ and $U(mn) \approx U(m) \oplus U(n)$.

F. Kernel: If $\phi : G \to \overline{G}$ is a group homomorphism, $\mathrm{Ker}(\phi) = \{x \in G \mid \phi(x) = e_{\overline{G}}\}$, where $e_{\overline{G}}$ is the identity element of $\overline{G}$.

G. Sylow's Theorem: For a finite group G and a prime p, let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of G. Then $\mathrm{Syl}_p(G) \neq \emptyset$. Let $P \in \mathrm{Syl}_p(G)$. Then $|\mathrm{Syl}_p(G)| = |G : N(P)| \equiv 1 \pmod p$, where $N(P) = \{x \in G \mid xPx^{-1} = P\}$.

1. Let N be a subgroup of a group G such that $xg = gx$ for all $x \in N$ and $g \in G$. (10 pts)

   (a) Show that $N \lhd G$.

   (b) Show that if G/N is cyclic, then G is Abelian.

ID#: **Name:**

2. Let H and K be subgroups of a group G. Show the following. (25 pts)

   (a) For $x \in G$, $xH = H$ if and only if $x \in H$.

   (b) $HH^{-1} = H$.

   (c) If $xhx^{-1} \in H$ for all $x \in G$ and $h \in H$, then H is a normal subgroup of G.

   (d) If H is a normal subgroup of G, then HK is a subgroup of G.

   (e) If $\alpha : G \to \overline{G}$ is a group homomorphism, then $|\alpha(x)| \mid |x|$ for every $x \in G$ of finite order.

ID#: **Name:**

3. Let G be a finite group and H a subgroup of G such that $|G:H| = n$. For each $g \in G$ let $\alpha_g : G/H \to G/H$ $(xH \mapsto gxH)$. (20 pts)

   (a) Show that $\alpha_g \in \mathrm{Sym}(G/H)$, i.e, α_g is a permutation on G/H.

   (b) Show that $\phi : G \to \mathrm{Sym}(G/H)$ $(g \mapsto \alpha_g)$ is a group homomorphism.

   (c) Show that $\mathrm{Ker}\phi = \bigcap_{x \in G} xHx^{-1}$.
   (Hint: $gxH = xH \Leftrightarrow gxHx^{-1} = xHx^{-1}$ and $xHx^{-1} \leq G$.)

   (d) G has a normal subgroup N such that $N \leq H$ and that $|G/N| \mid n!$.

ID#: **Name:**

4. Answer the following questions on Abelian groups of order $80 = 2^4 \cdot 5$. (20 pts)

   (a) Using the Fundamental Theorem of Finite Abelian Groups and list all non-isomorphic Abelian groups of order 80 and give a brief explanation.

   (b) List all Abelian groups of order 80 in your list above that have exactly three elements of order 2. Give your reason.

   (c) Determine whether or not $U(200) \approx U(220)$. Give your reason.

ID#: **Name:**

5. Let G be a group of order 60, P a Sylow 2-subgroup, Q a Sylow 3-subgroup and R a Sylow 5-subgroup of G. Suppose that $\{e\}$ and G are the only normal subgroups of G. Prove the following. (25 pts)

   (a) Show that P is Abelian.

   (b) Show that Q and R are cyclic.

   (c) Show that there are exactly 6 Sylow 5-subgroups and $|N(R)| = 10$.

   (d) Let H be a proper subgroup of G containing P. Then $|H| = 4$ or 12.

   (e) $G \approx A_5$. (Hint: Show that G has a subgroup of order 12 and use 3.)

Please write your message: Comments on group theory. Suggestions for improvements of this course. Write on the back of this sheet is also welcome.

Algebra I: Solutions to Final 2012 *June 22, 2012*

1. Let N be a subgroup of a group G such that $xg = gx$ for all $x \in N$ and $g \in G$. (10 pts)

   (a) Show that $N \triangleleft G$.
   Soln. Let $g \in G$. Then by assumption,
   $$gNg^{-1} = \{gxg^{-1} \mid x \in N\} = \{x \mid x \in N\} = N.$$
   Hence $N \triangleleft G$. ■

   (b) Show that if G/N is cyclic, then G is Abelian.
   Soln. Let $G/N = \langle gN \rangle = \{(gN)^n \mid n \in \boldsymbol{Z}\} = \{g^n N \mid n \in \boldsymbol{Z}\}$. Let $a, b \in G$. Then there exist $n, m \in \boldsymbol{Z}$ such that $a \in g^n N$ and $b \in g^m N$. Hence there exist $x, y \in N$ such that $a = g^n x$, $b = g^m y$. Now using assumption, we have
   $$ab = g^n x g^m y = g^n g^m xy = g^m g^n yx = g^m y g^n x = ba.$$
   Thus G is Abelian. ■

2. Let H and K be subgroups of a group G. Show the following. (25 pts)

   (a) For $x \in G$, $xH = H$ if and only if $x \in H$.
   Soln. Suppose $xH = H$. Since H is a subgroup, $e \in H$. Hence $x = xe \in xH = H$. Thus $x \in H$. Conversely, if $x \in H$, then since H is a subgroup,
   $$xH \subseteq HH \subseteq H = eH = xx^{-1}H \subseteq xHH \subseteq xH.$$
   Therefore $xH = H$. ■

   (b) $HH^{-1} = H$.
   Soln. Since H is a subgroup,
   $$H = He^{-1} \subseteq HH^{-1} \subseteq H.$$
   Therefore $H = HH^{-1}$. (One can use (a) as well. $HH^{-1} = \bigcup_{h \in H} Hh^{-1} = \bigcup_{h \in H} H = H$.) ■

   (c) If $xhx^{-1} \in H$ for all $x \in G$ and $h \in H$, then H is a normal subgroup of G.
   Soln. By assumption, $xHx^{-1} \subseteq H$ for every $x \in G$. Since $x^{-1} \in G$, $x^{-1}Hx = x^{-1}H(x^{-1})^{-1} \subseteq H$. Therefore
   $$xHx^{-1} \subseteq H = x(x^{-1}Hx)x^{-1} \subseteq xHx^{-1}.$$
   Thus $xHx^{-1} = H$ for every $x \in G$, and H is a normal subgroup of G. ■

   (d) If H is a normal subgroup of G, then HK is a subgroup of G.
   Soln. Suppose H is a normal subgroup of G. Then $e = ee \in HK$ and $HK \neq \emptyset$. Let $x, y \in HK$. Then there exist $h, h' \in H$ and $k, k' \in K$ such that $x = hk$ and $y = h'k'$. Since H is normal, $h'' = kh'k^{-1} \in H$ and
   $$xy = hkh'k' = h(kh'k^{-1})kk' = hh''kk' \in HK.$$
   Similarly since $h''' = k^{-1}h^{-1}k \in k^{-1}H(k^{-1})^{-1} = H$,
   $$x^{-1} = (hk)^{-1} = k^{-1}h^{-1} = (k^{-1}h^{-1}k)k^{-1} = h'''k^{-1} \in HK.$$
   Therefore HK is a subgroup of G. ■

(e) If $\alpha : G \to \overline{G}$ is a group homomorphism, then $|\alpha(x)| \mid |x|$ for every $x \in G$ of finite order.

Soln. Firstly, since $\alpha(e) = \alpha(ee) = \alpha(e)\alpha(e)$, we have $e = \alpha(e)$ by multiplying $\alpha(e)^{-1}$. Let $n = |x|$. Then $x^n = e$. So $e = \alpha(e) = \alpha(x^n) = \alpha(x)^n$. Thus $|\alpha(x)| \mid n$ by **B**. ■

3. Let G be a finite group and H a subgroup of G such that $|G : H| = n$. For each $g \in G$ let $\alpha_g : G/H \to G/H$ $(xH \mapsto gxH)$. (20 pts)

(a) Show that $\alpha_g \in \mathrm{Sym}(G/H)$, i.e, α_g is a permutation on G/H.

Soln. Since $\alpha_g(xH) = \alpha_g(yH)$ implies $gxH = gyH$ and $xH = yH$, α_g is one-to-one. Since G/H is a finite set, α_g is a bijection and $\alpha_g \in \mathrm{Sym}(G/H)$. ■

(b) Show that $\phi : G \to \mathrm{Sym}(G/H)$ $(g \mapsto \alpha_g)$ is a group homomorphism.

Soln. Since $\phi(gg') = \alpha_{gg'}$ and $\phi(g)\phi(g') = \alpha_g\alpha_{g'}$, we need to show $\alpha_{gg'} = \alpha_g\alpha_{g'}$ in $\mathrm{Sym}(G/H)$. This holds as

$$\alpha_{gg'}(xH) = gg'xH = g(g'xH) = \alpha_g(\alpha_{g'}(xH)) = (\alpha_g\alpha_{g'})(xH).$$

Thus ϕ is a group homomemorphism. ■

(c) Show that $\mathrm{Ker}\phi = \bigcap_{x \in G} xHx^{-1}$.

(Hint: $gxH = xH \Leftrightarrow gxHx^{-1} = xHx^{-1}$ and $xHx^{-1} \le G$.)

Soln. $g \in \mathrm{Ker}\phi$ if and only if $\alpha_g = id$ if and only if $gxH = xH$ for all $x \in G$. Since $gxH = xH \Leftrightarrow gxHx^{-1} = xHx^{-1}$ and xHx^{-1} is a subgroup of G, we have $g \in xHx^{-1}$. Thus

$$g \in \mathrm{Ker}\phi \Leftrightarrow (\forall x \in G)[g \in xHx^{-1}] \Leftrightarrow g \in \bigcap_{x \in G} xHx^{-1}.$$

Therefore we have the assertion. ■

(d) G has a normal subgroup N such that $N \le H$ and that $|G/N| \mid n!$.

Soln. Since $|G : H| = n$, $\mathrm{Sym}(G/H) \approx S_n$. Let $N = \mathrm{Ker}\phi \le H$. By first isomorphism theorem, G/N is isomorphic to a subgroup of $\mathrm{Sym}(G/H)$ and $|\mathrm{Sym}(G/H)| = n!$. Therefore $|G/N| \mid n!$. ■

4. Answer the following questions on Abelian groups of order $80 = 2^4 \cdot 5$. (20 pts)

(a) Using the Fundamental Theorem of Finite Abelian Groups and list all non-isomorphic Abelian groups of order 80 and give a brief explanation.

Soln. Since $4 = 4, 1+3, 2+2, 1+1+2, 1+1+1+1$, there are five isomorphism classes of Abelian groups of order $80 = 2^4 \cdot 5$. They are

$$\boldsymbol{Z}_{80},\ \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{40},\ \boldsymbol{Z}_4 \oplus \boldsymbol{Z}_{20},\ \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{20},\ \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{10}.$$

(b) List all Abelian groups of order 80 in your list above that have exactly three elements of order 2. Give your reason.

Soln. For each of the Abelian groups above, elements of order 2 are in

$$\boldsymbol{Z}_2,\ \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2,\ \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2,\ \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2,\ \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2.$$

Hence $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{40}$ and $\boldsymbol{Z}_4 \oplus \boldsymbol{Z}_{20}$ are those having three elements of order 2. ■

(c) Determine whether or not $U(200) \approx U(220)$. Give your reason.

Soln. $U(200) = U(2^3 \cdot 5^2) \approx U(2^3) \oplus U(5^2) \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{20}$ as all nonidentity elements of $U(2^3)$ are of order 2, and $U(5^2)$ is generated by 2. (Since the order of $U(5^2)$ is 20, it suffices to show the existence of an element of order divisible by 4. $7^2 \equiv -1 \pmod{25}$. So the order of 7 is four.)

$U(220) = U(2^2 \cdot 5 \cdot 11) \approx U(2^2) \oplus U(5) \oplus U(11) \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_4 \oplus \boldsymbol{Z}_{10} \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{20}$.

Hence these groups are isomorphic. ■

5. Let G be a group of order 60, P a Sylow 2-subgroup, Q a Sylow 3-subgroup and R a Sylow 5-subgroup of G. Suppose that $\{e\}$ and G are the only normal subgroups of G. Prove the following. (25 pts)

(a) Show that P is Abelian.

Soln. P is of order 4. Let $a \in P$ be a nonidentity element. Then $|a| = 2, 4$. If there is an element of order 4, P is cyclic and P is Abelian. Hence we may assume that $x^2 = e$ for every $x \in P$. For $x, y \in P$, $xy = xy(yx)^2 = (x(yy)x)yx = yx$ and P is Abelian. ■

(b) Show that Q and R are cyclic.

Soln. Q and R are of prime order. Let $x \in Q$ and $y \in R$ be nonidentity elements. Then $1 \neq |x| \mid 3$ and $1 \neq |y| \mid 5$, and $|x| = 3$, $|y| = 5$, Therefore $Q = \langle x \rangle$ and $R = \langle y \rangle$ are both cyclic. ■

(c) Show that there are exactly 6 Sylow 5-subgroups and $|N(R)| = 10$.

Soln. $\mathrm{Syl}_5(G)| \equiv 1 \pmod{5}$ are divisors of $|G| = 60$, we have $|\mathrm{Syl}_5(G)| = 1$ or 6. If $|\mathrm{Syl}_5(G)| = 1$, R is normal. This contradicts our assumption. Hence $6 = |\mathrm{Syl}_5(G)| = |G : N(R)|$, and $|N(R)| = 10$ by **C**. ■

(d) Let H be a proper subgroup of G containing P. Then $|H| = 4$ or 12.

Soln. Since $4 \mid |H|$, we need to show that $|H| \neq 20$. Suppose $|H| = 20$ and $R \leq H$. Then $|\mathrm{Syl}_5(H)| = 1$. Thus $R \lhd H$, and $|N(R)|$ is divisible by 4. This contradicts (c). ■

(e) $G \approx A_5$. (Hint: Show that G has a subgroup of order 12 and use 3.)

Soln. Suppose $|N(P)| = 4$, i.e., $N(P) = P$. Let $g \in G - P$. Suppose $e \neq \exists z \in P \cap gPg^{-1}$. Then $C(z)$ contains both P and gPg^{-1} and so $C(z)$ is a subgroup properly containing P. This contradicts our assumption. Hence $P \cap gPg^{-1} = \{e\}$ for all $g \notin P$. This is absurd as $\bigcup_{g \in G} gPg^{-1}$ must contain 46 elements, while there are 24 elements of order 5. Thus G has a subgroup H of order 12. Since $|G : H| = 5$, G is isomorphic to a subgroup $\overline{G}$ of S_5. Since $A_5 \lhd S_5$, $\overline{G} \cap A_5 \lhd \overline{G}$ and $\overline{G} = A_5$. Note that every Sylow 5-subgoup of S_5 is in A_5 and hence $\overline{G} \cap A_5 \neq \{e\}$. This proves the assertion. ■