Algebra I: Final 2013

June 24, 2013

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Quote the following when necessary.

A. Subgroup H of a group G:

$$H \leq G \Leftrightarrow \emptyset \neq H \subseteq G,\ xy \in H \text{ and } x^{-1} \in H \text{ for all } x, y \in H.$$

B. Order of an Element: Let g be an element of a group G. Then $\langle g \rangle = \{g^n \mid n \in \boldsymbol{Z}\}$ is a subgroup of G. If there is a positive integer m such that $g^m = e$, where e is the identity element of G, $|g| = \min\{m \mid g^m = e,\ m \in \boldsymbol{N}\}$ and $|g| = |\langle g \rangle|$. Moreover, for any integer n, $|g|$ divides n if and only if $g^n = e$.

C. Lagrange's Theorem: If H is a subgroup of a finite group G, then $|G| = |G : H||H|$.

D. Normal Subgroup: A subgroup H of a group G is normal if $gHg^{-1} = H$ for all $g \in G$. If H is a normal subgroup of G, then G/H becomes a group with respect to the binary operation $(gH)(g'H) = gg'H$.

E. Direct Product: If $\gcd\{m, n\} = 1$, then $\boldsymbol{Z}_{mn} \approx \boldsymbol{Z}_m \oplus \boldsymbol{Z}_n$ and $U(mn) \approx U(m) \oplus U(n)$.

F. Isomorphism Theorem: If $\alpha : G \to \overline{G}$ is a group homomorphism, $\mathrm{Ker}(\alpha) = \{x \in G \mid \alpha(x) = e_{\overline{G}}\}$, where $e_{\overline{G}}$ is the identity element of $\overline{G}$. Then $\alpha(G) \leq \overline{G}$, $\mathrm{Ker}(\alpha)$ is a normal subgroup of G, and $G/\mathrm{Ker}(\alpha) \approx \alpha(G)$.

G. Sylow's Theorem: For a finite group G and a prime p, let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of G. Then $\mathrm{Syl}_p(G) \neq \emptyset$. Let $P \in \mathrm{Syl}_p(G)$. Then $|\mathrm{Syl}_p(G)| = |G : N(P)| \equiv 1 \pmod{p}$, where $N(P) = \{x \in G \mid xPx^{-1} = P\}$.

Other Theorems: List other theorems you applied in your solutions.

1. Let H be a subgroup of a group G. Let $a, b \in G$. Show the following. (10 pts)

   (a) $aH = bH$ if and only if $a^{-1}b \in H$.

   (b) If $aH \cap bH \neq \emptyset$, then $aH = bH$.

2. Let $\alpha : G \to A$ be an onto group homomorphism, and B a normal subgroup of A. Show the following. (10 pts)

   (a) $\alpha^{-1}(B) = \{x \in G \mid \alpha(x) \in B\}$ is a normal subgroup of G.

   (b) $G/\alpha^{-1}(B) \approx A/B$.

3. Let H be a normal subgroup of a group G. Show the following. (20 pts)

   (a) For $x \in G$, let $\phi_x : H \to H$ $(h \mapsto xhx^{-1})$. Then $\phi_x \in \mathrm{Aut}(H)$, i.e., ϕ_x is a bijective homomorphism from H to H.

   (b) Let $\Phi : G \to \mathrm{Aut}(H)$ $(x \mapsto \phi_x)$. Then Φ is a (group) homomorphism.

   (c) Let $C(H) = \{x \in G \mid xh = hx \text{ for all } h \in H\}$. Then $C(H) \lhd G$ and $G/C(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

   (d) If H is cyclic, then $G/C(H)$ is Abelian.

4. Answer the following questions on Abelian groups of order $32 = 2^5$. (20 pts)

(a) Using the Fundamental Theorem of Finite Abelian Groups and list all non-isomorphic Abelian groups of order 32 and give a brief explanation.

(b) List all Abelian groups of order 32 in your list in (a) that have exactly seven elements of order 2. Give your reason.

(c) Express $U(5 \cdot 16)$ as an internal direct product of cyclic subgroups, and identify a group isomorphic to $U(5 \cdot 16)$ in your list in (a).

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5. Let G be a group and H a subgroup of G. Show the following. (20 pts)

   (a) For $x \in G$, $xHx^{-1} \leq G$.

   (b) Suppose for some $x \in G$, $G = H(xHx^{-1})$. Then $G = H$. (Hint: Express x^{-1} as an element of $H(xHx^{-1})$.)

6. Let p be a prime and P a group of order p^2. Show the following.

   (a) Let Q be a subgroup of P of order p. Then $Q \lhd P$.

   (b) P is Abelian.

7. Let p and q are distinct primes. Let G be a group of order p^2q. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Show the following. (20 pts)

   (a) If $Q \not\triangleleft G$, then $|\mathrm{Syl}_q(G)| = p$ or p^2.

   (b) If $|\mathrm{Syl}_q(G)| = p^2$, then $P \triangleleft G$.

   (c) If $|\mathrm{Syl}_q(G)| = p$, then $p > q$ and $P \triangleleft G$.

   (d) Find an example of a group G satisfying $|\mathrm{Syl}_q(G)| = p$.

Please write your message: Comments on group theory. Suggestions for improvements of this course. Write on the back of this sheet is also welcome.

Algebra I: Solutions to Final 2013

June 24, 2013

1. Let H be a subgroup of a group G. Let $a, b \in G$. Show the following. (10 pts)

   (a) $aH = bH$ if and only if $a^{-1}b \in H$.

   Soln. Since $H \leq G$, $H \neq \emptyset$. Let $a \in H$. Then $a^{-1} \in H$ and $e = aa^{-1} \in H$.

   Suppose $aH = bH$. Since $e \in H$, $aH = bH$ implies that $b = be \in bH = aH$. Hence there exists $h \in H$ such that $b = ah$. Therefore by multiplying b^{-1} to both hand sides from the left, $b^{-1}a = h \in H$.

   Conversely let $a^{-1}b = h \in H$. Then $b = ah$ and

   $$bH = ahH \subset aH = aeH = ahh^{-1}H = aa^{-1}bh^{-1}H \subset bH.$$

   Therefore $aH = bH$. ■

   (b) If $aH \cap bH \neq \emptyset$, then $aH = bH$.

   Soln. Let $c \in aH \cap bH$. Then $c = ah = bh'$ for some $h, h' \in H$. So $a^{-1}c = h \in H$ and $b^{-1}c = h' \in H$. Hence by (a), $aH = cH = bH$. ■

2. Let $\alpha : G \to A$ be an onto group homomorphism, and B a normal subgroup of A. Show the following. (10 pts)

   (a) $\alpha^{-1}(B) = \{x \in G \mid \alpha(x) \in B\}$ is a normal subgroup of G.

   Soln. Let $H = \alpha^{-1}(B)$. We show $H \leq G$ by one step subgroup test. For $x, y \in H$, $\alpha(x), \alpha(y) \in B$. Hence $\alpha(x^{-1}y) = \alpha(x)^{-1}\alpha(y) \in B$ and $x^{-1}y \in H$. Therefore $H \leq G$.

   Let $h \in H$ and $x \in G$. Since B is a normal subgroup of A,

   $$\alpha(xhx^{-1}) = \alpha(x)\alpha(h)\alpha(x)^{-1} \in \alpha(x)B\alpha(x)^{-1} \subset B.$$

   Therefore $xhx^{-1} \in H$ amd $xHx^{-1} \subset H$. Since x is arbitrary, $x^{-1}Hx = x^{-1}H(x^{-1})^{-1} \subset H$. So by multiplying x from the left and x^{-1} from the right, we have $H \subset xHx^{-1}$. Therefore $xHx^{-1} = H$ for all $x \in G$ and $H \lhd G$. ■

   (b) $G/\alpha^{-1}(B) \approx A/B$.

   Soln. Since α is an onto homomorphism, $\beta : G \to A/B$ $(x \mapsto xB)$ is an onto homomorphism as well. Since the kernel is $\alpha^{-1}(B)$, we have $G/\alpha^{-1}(B) \approx A/B$ by Isomorphism Theorem. ■

3. Let H be a normal subgroup of a group G. Show the following. (20 pts)

   (a) For $x \in G$, let $\phi_x : H \to H$ $(h \mapsto xhx^{-1})$. Then $\phi_x \in \mathrm{Aut}(H)$, i.e., ϕ_x is a bijective homomorphism from H to H.

   Soln. Since $H \lhd G$, $xhx^{-1} \in xHx^{-1} = H$. ϕ_x is onto as $x^{-1}hx \in H$ for $h \in H$, and $\phi_x(x^{-1}hx) = xx^{-1}hxx^{-1} = h$. ϕ_x is one to one as $\phi_x(h) = \phi_x(h')$ implies, $xhx^{-1} = xh'x^{-1}$ and $h = h'$. ϕ_x is a homomorphism as $\phi_x(hh') = xhhx^{-1} = xhx^{-1}xhx^{-1} = \phi_x(h)\phi_x(h')$. Therefore $\phi_x \in \mathrm{Aut}(H)$. ■

(b) Let $\Phi : G \to \text{Aut}(H)$ $(x \mapsto \phi_x)$. Then Φ is a (group) homomorphism.

Soln. $\Phi(xy) = \phi_{xy}$ and $\Phi(x)\Phi(y) = \phi_x\phi_y$. Hence it suffices to show that $\phi_{xy} = \phi_x\phi_y$ in $\text{Aut}(H)$. For $h \in H$,

$$\phi_{xy}(h) = xyh(xy)^{-1} = x(yhy^{-1})x^{-1} = \phi_x(yhy^{-1}) = \phi_x(\phi_y(h)) = (\phi_x\phi_y)(h),$$

as desired. ■

(c) Let $C(H) = \{x \in G \mid xh = hx \text{ for all } h \in H\}$. Then $C(H) \lhd G$ and $G/C(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$.

Soln. $\text{Ker}(\Phi) = \{x \in G \mid \phi_x = id_H\}$, and $\phi_x = id_H$ if and only if $xhx^{-1} = h$ for all $h \in H$. Thus $\text{Ker}(\Phi) = C(H)$. Since $\text{Ker}\Phi$ is a normal subgroup in G by Problem 2(a), $C(H) \lhd G$. ■

(d) If H is cyclic, then $G/C(H)$ is Abelian.

Soln. Since $G/C(H)$ is isormophic to a subgroup of $\text{Aut}(H)$ by Isomoprhism Theorem, it suffices to show that $\text{Aut}(H)$ is Abelian when H is cyclic. Let $H = \langle x \rangle$, and $\sigma \in \text{Aut}(H)$. Then $\sigma(x^n) = \sigma(x)^n$ for all $n \in \boldsymbol{Z}$. Hence σ is determined by $\sigma(x)$. Suppose $\sigma, \tau \in \text{Aut}(H)$ with $\sigma(x) = x^i$ and $\tau(x) = x^j$. Then

$$(\sigma\tau)(x) = \sigma(\tau(x)) = \sigma(x^j) = \sigma(x)^j = x^{ij} = \tau(x)^i = \tau(x^i) = \tau(\sigma(x)) = (\tau\sigma)(x).$$

Therefore $\sigma\tau = \tau\sigma$. ■

4. Answer the following questions on Abelian groups of order $32 = 2^5$. (20 pts)

(a) Using the Fundamental Theorem of Finite Abelian Groups and list all non-isomorphic Abelian groups of order 32 and give a brief explanation.

Soln. Since every finite Abelian group is isomorphic to an external direct product of cyclic groups, and it can be written uniquely as $\boldsymbol{Z}_{e_1} \oplus \boldsymbol{Z}_{e_2} \oplus \cdots \oplus \boldsymbol{Z}_{e_r}$ with $e_1 \mid e_2, e_2 \mid e_3, \ldots, e_{r-1} \mid e_r$, which is called of type $(e_1, e_2, \ldots, e_r)$. Therefore we have

(32): $\boldsymbol{Z}_{32}$
(2,16): $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{16}$
(4,8): $\boldsymbol{Z}_4 \oplus \boldsymbol{Z}_8$
(2,2,8): $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_8$
(2,4,4): $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_4 \oplus \boldsymbol{Z}_4$
(2,2,2,4): $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_4$
(2,2,2,2,2): $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$

(b) List all Abelian groups of order 32 in your list in (a) that have exactly seven elements of order 2. Give your reason.

Soln. Seven elements of order 2 form a group isormopic to $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$, they are $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_8$ or $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_4 \oplus \boldsymbol{Z}_4$. ■

(c) Express $U(5 \cdot 16)$ as an <u>internal</u> direct product of <u>cyclic</u> subgroups, and identify a group isomorphic to $U(5 \cdot 16)$ in your list in (a).

Soln. $U(5 \cdot 16) \approx U_{16}(5 \cdot 16) \oplus U_5(5 \cdot 16)$, and $\langle 17 \rangle = U_{16}(5 \cdot 16) \approx U(5) \approx \boldsymbol{Z}_4$, $U_5(5 \cdot 16) \approx U(16) \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_4$.

$$U_{16}(5 \cdot 16) = \{1, 17, 33, 49\} = \langle 17 \rangle = \langle 33 \rangle \approx \boldsymbol{Z}_4.$$

$$U_5(5 \cdot 16) = \{1, 11, 21, 31, 41, 51, 61, 71\} = \langle 31 \rangle \times \langle 11 \rangle \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_4.$$

Therefore,

$$U(5 \cdot 16) = \langle 31 \rangle \times \langle 11 \rangle \times \langle 33 \rangle \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_4 \oplus \boldsymbol{Z}_4$$

5. Let G be a group and H a subgroup of G. Show the following. (20 pts)

   (a) For $x \in G$, $xHx^{-1} \leq G$.

   Soln. For $h, h' \in H$, $xhx^{-1}xh'x^{-1} = xhh'x^{-1} \in xHx^{-1}$ and $(xhx^{-1})^{-1} = xh^{-1}x^{-1} \in xHx^{-1}$. Therefore $xHx^{-1} \leq G$. ■

   (b) Suppose for some $x \in G$, $G = H(xHx^{-1})$. Then $G = H$. (Hint: Express x^{-1} as an element of $H(xHx^{-1})$.)

   Soln. Suppose $x^{-1} = hxh'x^{-1}$ for some $h, h' \in H$. Then $hxh' = e$ and $x = h^{-1}h'^{-1} \in H$. Therefore $xHx^{-1} = H$, and $G = H$. ■

6. Let p be a prime and P a group of order p^2. Show the following.

   (a) Let Q be a subgroup of P of order p. Then $Q \lhd P$.

   Soln. Suppose Q is not normal in P. Then there exists $x \in G$ such that $Q \neq xQx^{-1}$. Since $Q \cap xQx^{-1} = \{e\}$, $QxQx^{-1} = P$. This contradicts Problem 5 (b). So Q is normal. Note that $QxQx^{-1} = P$ is because if $Q = \langle y \rangle$, $y^i xQx^{-1} \neq y^j xQx^{-1}$ unless $y^{i-j} = e$, i.e., $y^i = y^j$ by Problem 1 (a) and Problem 5 (a). ■

   (b) P is Abelian.

   Soln. We may assume that P is not cyclic. Hence every nonidentity element of P generates a cyclic subgroup of order p by (C). Let Q be a subgoup of order p, and $x \notin Q$. Then x is of order p again, as P is not cyclic. Let $R = \langle x \rangle$. Then both Q and R are normal and $Q \cap R = \{e\}$ as $x \notin Q$. Threfore $QR = Q \times R \leq P$. By comparing their orders, we have $P = Q \times R$ and P is Abelian. ■

7. Let p and q are distinct primes. Let G be a group of order p^2q. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Show the following. (20 pts)

   (a) If $Q \not\lhd G$, then $|\mathrm{Syl}_q(G)| = p$ or p^2.

   Soln. Since $p = |\mathrm{Syl}_q(G)| = |G : N(Q)|$ and $Q \leq N(Q)$. By (C), $|G : N(Q)| \mid p^2$. Moreover $|G : N(Q)| = 1$ if and only if $N(Q) = G$ and $Q \lhd G$. Hence by our assumption, we have the conclusion. ■

   (b) If $|\mathrm{Syl}_q(G)| = p^2$, then $P \lhd G$.

   Soln. Since there are $q - 1$ elements of order q in a Sylow q-subgroup of G, there are $p^2(q-1)$ elements of order q in G in this case. There are only p^2 remaining elements. There are no elements of order q in a Sylow p subgroup of G, which is of order p^2, P is the unique Sylow p-subgroup and $P \lhd G$.

   (c) If $|\mathrm{Syl}_q(G)| = p$, then $p > q$ and $P \lhd G$.

   Soln. Since $|\mathrm{Syl}_q(G)| \equiv 1 \pmod{q}$, $q \mid p - 1$. Thus $q < p$. Since $|\mathrm{Syl}_p(G)| \equiv 1 \pmod{p}$ and this number divides q, $|\mathrm{Syl}_p(G)| = 1$ as $p > q$ and p does not divide $q - 1$. Therefore, $P \lhd G$. ■

   (d) Find an example of a group G satisfying $|\mathrm{Syl}_q(G)| = p$.

   Soln. $G = \boldsymbol{Z}_3 \oplus S_3$, $p = 3$, $q = 2$. ■