Algebra I: Final 2014

June 20, 2014

ID#: **Name:**

Quote the following when necessary.

A. Subgroup H of a group G:

$$H \leq G \Leftrightarrow \emptyset \neq H \subseteq G,\ xy \in H \text{ and } x^{-1} \in H \text{ for all } x, y \in H.$$

B. Order of an Element: Let g be an element of a group G. Then $\langle g \rangle = \{g^n \mid n \in \boldsymbol{Z}\}$ is a subgroup of G. If there is a positive integer m such that $g^m = e$, where e is the identity element of G, $|g| = \min\{m \mid g^m = e,\ m \in \boldsymbol{N}\}$ and $|g| = |\langle g \rangle|$. Moreover, for any integer n, $|g|$ divides n if and only if $g^n = e$.

C. Lagrange's Theorem: If H is a subgroup of a finite group G, then $|G| = |G : H||H|$.

D. Normal Subgroup: A subgroup H of a group G is normal if $gHg^{-1} = H$ for all $g \in G$. If H is a normal subgroup of G, then G/H becomes a group with respect to the binary operation $(gH)(g'H) = gg'H$.

E. Direct Product: If $\gcd\{m, n\} = 1$, then $\boldsymbol{Z}_{mn} \approx \boldsymbol{Z}_m \oplus \boldsymbol{Z}_n$ and $U(mn) \approx U(m) \oplus U(n)$.

F. Isomorphism Theorem: If $\alpha : G \to \overline{G}$ is a group homomorphism, $\mathrm{Ker}(\alpha) = \{x \in G \mid \alpha(x) = e_{\overline{G}}\}$, where $e_{\overline{G}}$ is the identity element of $\overline{G}$. Then $\alpha(G) \leq \overline{G}$, $\mathrm{Ker}(\alpha)$ is a normal subgroup of G, and $G/\mathrm{Ker}(\alpha) \approx \alpha(G)$.

G. Sylow's Theorem: For a finite group G and a prime p, let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups of G. Then $\mathrm{Syl}_p(G) \neq \emptyset$. Let $P \in \mathrm{Syl}_p(G)$. Then $|\mathrm{Syl}_p(G)| = |G : N(P)| \equiv 1 \pmod{p}$, where $N(P) = \{x \in G \mid xPx^{-1} = P\}$.

Other Theorems: List other theorems you applied in your solutions.

1. Let H be a subgroup of a group G. Let $a, b \in G$. Show the following. (10 pts)

   (a) $aH = bH$ if and only if $a^{-1}b \in H$.

   (b) If $aH = Hb$, then $aH = bH$.

2. Let H and K be subgroups of G. Show the following. (10 pts)

   (a) If HK is a subgroup of G, then $HK = KH$.

   (b) If $hK = Kh$ for all $h \in H$, then HK is a subgroup of G.

3. Let $\phi : G \to H$ be an onto group homomorphism, e_G is the identity element of G and e_H the identity element of H. Show the following. (20 pts)

   (a) $\phi(e_G) = e_H$ and for $a \in G$, $\phi(a^{-1}) = \phi(a)^{-1}$.

   (b) $\text{Ker}\phi = \{x \in G \mid \phi(x) = e_H\}$ is a normal subgroup of G.

   (c) If G is cyclic, then H is cyclic.

   (d) If H is Abelian, then $\phi(aba^{-1}b^{-1}) = e_H$ for all $a, b \in G$.

4. Answer the following questions on Abelian groups of order $162 = 2 \cdot 3^4$. (20 pts)

   (a) Using the Fundamental Theorem of Finite Abelian Groups and list all non-isomorphic Abelian groups of order 162.

   (b) Explain that there is exactly one element of order 2.

   (c) Explain that if there is only one subgroup of order 3, then it is cyclic.

   (d) Let $G = \boldsymbol{Z}_{18} \oplus \boldsymbol{Z}_9$. Find the number of elements of order 3 and the number of subgroups of order 3.

5. Let $\boldsymbol{Q}^*$ be the multiplicative group of nonzero rational numbers, and $\boldsymbol{Q}$ the additive group of rational numbers. Show the following. (20 pts)

   (a) $\boldsymbol{Q}^*$ is not cyclic.

   (b) Let H be a finite subgroup of $\boldsymbol{Q}^*$. Then $H = \{1\}$ or $H = \{1, -1\}$.

   (c) $\boldsymbol{Q}$ is not isomorphic to $\boldsymbol{Q}^*$.

   (d) Let $\phi : \boldsymbol{Q} \to \boldsymbol{Q}$ be a group automorphism. Then $\phi(a) = a\phi(1)$ for every $a \in \boldsymbol{Q}$ and $\phi(1) \in \boldsymbol{Q}^*$.

6. Let G be a group of order $165 = 3 \cdot 5 \cdot 11$. Let $P \in \mathrm{Syl}_{11}(G)$ and $Q \in \mathrm{Syl}_5(G)$. Show the following. (20 pts)

   (a) P is a normal subgroup of G.

   (b) Suppose Q is not a normal subgroup. Let $H = N(Q) = \{x \in G \mid xQx^{-1} = Q\}$ and $R \in \mathrm{Syl}_3(N(Q))$.

      i. $|H| = 15$.

      ii. $H = Q \times R$.

   (c) $|\mathrm{Syl}_3(G)| = 1$.

Please write your message: Comments on group theory. Suggestions for improvements of this course. Write on the back of this sheet is also welcome.

Algebra I: Solutions to Final 2014

June 20, 2014

1. Let H be a subgroup of a group G. Let $a, b \in G$. Show the following. (10 pts)

   (a) $aH = bH$ if and only if $a^{-1}b \in H$.

   Soln. Since $H \leq G$, $H \neq \emptyset$. Let $a \in H$. Then $a^{-1} \in H$ and $e = aa^{-1} \in H$.

   Suppose $aH = bH$. Since $e \in H$, $aH = bH$ implies that $b = be \in bH = aH$. Hence there exists $h \in H$ such that $b = ah$. Therefore by multiplying a^{-1} to both hand sides from the left, $a^{-1}b = h \in H$.

   Conversely let $a^{-1}b = h \in H$. Then $b = ah$ and

   $$bH = ahH \subset aH = aeH = ahh^{-1}H = aa^{-1}bh^{-1}H \subset bH.$$

   Therefore $aH = bH$. ■

   (b) If $aH = Hb$, then $aH = bH$.

   Soln. Since $b = eb \in Hb = aH$, there is $h \in H$ such that $b = ah$. Hence $a^{-1}b \in H$. Therefore $aH = bH$ by (a). ■

2. Let H and K be subgroups of G. Show the following. (10 pts)

   (a) If HK is a subgroup of G, then $HK = KH$.

   Soln. Since $e \in H \cap K$, for $h \in H$ and $k \in K$, $h = he \in HK$ and $k = ek \in HK$. Since HK is a subgroup of G, and $h, k \in HK$, $kh \in HK$. Hence $KH \subset HK$. Since $(hk)^{-1} \in HK$, there exist $h' \in H$ and $k' \in K$ such that $(hk)^{-1} = h'k'$. Therefore, $hk = ((hk)^{-1})^{-1} = (h'k')^{-1} = k'^{-1}h'^{-1} \in KH$. Hence $HK \subset KH$. Therefore $HK = KH$. ■

   (b) If $hK = Kh$ for all $h \in H$, then HK is a subgroup of G.

   Soln. Let $h, h' \in H$ and $k, k' \in K$. Since $h'K = Kh' \ni kh'$, there is $k'' \in K$ such that $h'k'' = kh'$. Hence $hkh'k' = hh'k''k' \in HK$. Since $(hk)^{-1} = k^{-1}h^{-1} \in Kh^{-1} = h^{-1}K \subset HK$. Therefore, HK is a subgroup of G. ■

3. Let $\phi : G \to H$ be an onto group homomorphism, e_G is the identity element of G and e_H the identity element of H. Show the following. (20 pts)

   (a) $\phi(e_G) = e_H$ and for $a \in G$, $\phi(a^{-1}) = \phi(a)^{-1}$.

   Soln. $\phi(e_G) = \phi(e_G)^{-1}\phi(e_G)\phi(e_G) = \phi(e_G)^{-1}\phi(e_Ge_G) = \phi(e_G)^{-1}\phi(e_G) = e_H$. $\phi(a^{-1}) = \phi(a^{-1})\phi(a)\phi(a)^{-1} = \phi(a^{-1}a)\phi(a)^{-1} = \phi(e_G)\phi(a)^{-1} = e_H\phi(a)^{-1} = \phi(a)^{-1}$. ■

   (b) $\text{Ker}\phi = \{x \in G \mid \phi(x) = e_H\}$ is a normal subgroup of G.

   Soln. Let $a, b \in \text{Ker}\phi$. Then $\phi(ab) = \phi(a)\phi(b) = e_He_h = e_h$. Hence $ab \in \text{Ker}\phi$. By (a) $\phi(a^{-1}) = \phi(a)^{-1} = e_H^{-1} = e_H$. Hence $a^{-1} \in \text{Ker}\phi$. Thus $\text{Ker}\phi$ is a subgroup of G. Let $g \in G$, then $\phi(gag^{-1}) = \phi(g)\phi(a)\phi(g^{-1}) = \phi(g)e_H\phi(g)^{-1} = e_H$. Hence $g\text{Ker}\phi g^{-1} \subset \text{Ker}\phi$ for all $g \in G$. Since this holds for $g^{-1} \in G$, $g^{-1}\text{Ker}\phi g \subset \text{Ker}\phi$, which implies $\text{Ker}\phi \subset g\text{Ker}\phi g^{-1}$. Thus $g\text{Ker}\phi g^{-1} = \text{Ker}\phi$ for all $g \in G$ and $\text{Ker}\phi$ is a normal subgroup of G. ■

(c) If G is cyclic, then H is cyclic.

Soln. Suppose $G = \langle a \rangle = \{a^n \mid n \in \boldsymbol{Z}\}$. Let $b = \phi(a)$. Sincce ϕ is onto,

$$H = \phi(G) = \{\phi(a^n) \mid n \in \boldsymbol{Z}\} = \{\phi(a)^n \mid n \in \boldsymbol{Z}\} = \{b^n \mid n \in \boldsymbol{Z}\} = \langle b \rangle.$$

Therefore H is cyclic. ∎

(d) If H is Abelian, then $\phi(aba^{-1}b^{-1}) = e_H$ for all $a, b \in G$.

Soln. Since H is Abelian, $\phi(b)\phi(a)^{-1} = \phi(a)^{-1}\phi(b)$, it follows from (a) that

$$\phi(aba^{-1}b^{-1}) = \phi(a)\phi(b)\phi(a)^{-1}\phi(b)^{-1} = \phi(a)\phi(a)^{-1}\phi(b)\phi(b)^{-1} = e_H,$$

for all $a, b \in G$.

4. Answer the following questions on Abelian groups of order $162 = 2 \cdot 3^4$. (20 pts)

(a) Using the Fundamental Theorem of Finite Abelian Groups and list all non-isomorphic Abelian groups of order 162.

Soln.

i. $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{81} \approx \boldsymbol{Z}_{162}$.
ii. $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_{27} \approx \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_{54}$.
iii. $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_9 \oplus \boldsymbol{Z}_9 \approx \boldsymbol{Z}_9 \oplus \boldsymbol{Z}_{18}$.
iv. $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_9 \approx \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_{18}$.
v. $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \approx \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_6$.

(b) Explain that there is exactly one element of order 2.

Soln. By above every Abelian group G of order 162 can be written as an external direct sum of $\boldsymbol{Z}_2$ and a group H of oder 81, i.e., $G \approx \boldsymbol{Z}_2 \oplus H$, For $x = (a, b) \in \boldsymbol{Z}_2 \oplus H$, $|x| = \mathrm{lcm}\{|a|, |b|\}$. By Lagrange's theorem $|b|$ is a divisor of 81. Hence if x is of order 2, $x = (1, e_H)$ and there is exactly one element of order 2. ∎

(c) Explain that if there is only one subgroup of order 3, then it is cyclic.

Soln. The cases ii-v have 4, 4, 13, 40 subgroups of order 3. Hence the only possibility is the first case, which is cyclic. ∎

(d) Let $G = \boldsymbol{Z}_{18} \oplus \boldsymbol{Z}_9$. Find the number of elements of order 3 and the number of subgroups of order 3.

Soln. There are $3^2 - 1$ elements of order 3, and 4 subgroups of order 3 as each subgroup of order 3 contains two elements of order 3. ∎

5. Let $\boldsymbol{Q}^*$ be the multiplicative group of nonzero rational numbers, and $\boldsymbol{Q}$ the additive group of rational numbers. Show the following. (20 pts)

(a) $\boldsymbol{Q}^*$ is not cyclic.

Soln. Suppose $\boldsymbol{Q}^* = \langle a \rangle$. Let $a = m/n$ with $m, n \in \boldsymbol{Z}$ such that $\gcd\{m.n\} = 1$. There is a prime p such that p is coprime to m and n.. If $p = a^k$ with $k \geq 0$, then $pn^k = m^k$ and $p \mid m$, a contradiction. If $p = a^k$ with $k < 0$, then $pm^{-k} = n^{-k}$ and $p \mid n$, a contradiction. Therefore, $\boldsymbol{Q}^*$ is not cyclic. ∎

(b) Let H be a finite subgroup of $\boldsymbol{Q}^*$. Then $H = \{1\}$ or $H = \{1, -1\}$.

Soln. Let $x \in H$. Then $x^n = 1$ for some positive integer n. Since x is a real number and $|x| = 1$, $x = 1$ or -1. Therefore, $H = \{1\}$ or $H = \{1, -1\}$.

(c) $\boldsymbol{Q}$ is not isomorphic to $\boldsymbol{Q}^*$.

Soln. Let $a \in \boldsymbol{Q}$, then $na = 0$ implies $n = 0$ or $a = 0$. Hence there are no elements of order 2 in $\boldsymbol{Q}$. Since -1 is an element of order 2 in $\boldsymbol{Q}^*$, $\boldsymbol{Q}$ is not isomorphic to $\boldsymbol{Q}^*$. ■

(d) Let $\phi : \boldsymbol{Q} \to \boldsymbol{Q}$ be a group automorphism. Then $\phi(a) = a\phi(1)$ for every $a \in \boldsymbol{Q}$ and $\phi(1) \in \boldsymbol{Q}^*$.

Soln. If n is an integer $\phi(n) = \phi(n1) = n\phi(1)$, as ϕ is an additive group homomorphism. If m is a positive integer, $\phi(1) = \phi(m(1/m)) = m\phi(1/m)$. Hence $\phi(1/m) = (1/m)\phi(1)$. Let $a = n/m \in \boldsymbol{Q}$ with $m, n \in \boldsymbol{Z}$ and $m \neq 0$. Then $\phi(a) = \phi(n/m) = n\phi(1/m) = (n/m)\phi(1) = a\phi(1)$. If $\phi(1) = 0$, then $\phi(a) = 0$ for all $a \in \boldsymbol{Q}$. Since ϕ is a group automorphism and hence onto, $\phi(1) \neq 0$. ■

6. Let G be a group of order $165 = 3 \cdot 5 \cdot 11$. Let $P \in \mathrm{Syl}_{11}(G)$ and $Q \in \mathrm{Syl}_5(G)$. Show the following. (20 pts)

(a) P is a normal subgroup of G.

Soln. By Sylow's Theorem, $|\mathrm{Syl}_{11}(G)| = |G : N(P)| \equiv 1 \pmod{11}$. Since $|G : N(P)|$ is a divisor of $|G|$, the only possibility is 1. Therefore $G = N(P)$. Since $N(P) = \{x \in G \mid xPx^{-1} = P\}$, $N(P) = G$ implies P is normal in G. ■

(b) Suppose Q is not a normal subgroup. Let $H = N(Q) = \{x \in G \mid xQx^{-1} = Q\}$ and $R \in \mathrm{Syl}_3(N(Q))$.

i. $|H| = 15$.

Soln. By Sylow's Theorem, $|\mathrm{Syl}_5(G)| = |G : N(Q)| \equiv 1 \pmod 5$. Since $|G : N(Q)|$ is a divisor of $|G|$, the possibilities are 1 and 11. If it is 1, Q is normal. Hence $|G : N(Q)| = 11$, and $|H| = |N(Q)| = 15$. ■

ii. $H = Q \times R$.

Soln. Since $H = N(Q)$, Q is normal in H. $|\mathrm{Syl}_3(H)| = |H : N_H(R)| \equiv 1 \pmod 3$, where $N_H(R) = H \cap N(R)$. Since $|H| = 15$, the number is 1 and $H = N_H(R)$. Therefore R is normal in H. Since $|Q| = 5$ and $|R| = 3$, $|Q \cap R| = 1$ and $H = Q \times R$. Note that by Problem 2 (b), QR is a subgroup of H and $QR = Q \cap R$ is of order 15. Hence $H = QR = Q \times R$. ■

This part shows that a group of order 15 *is always cyclic and its Sylow subgroups are normal in the group.*

(c) $|\mathrm{Syl}_3(G)| = 1$.

Soln. Let R be a Sylow 3-subgroup. Suppose $|\mathrm{Syl}_3(G)| \neq 1$. Then $|G : N(R)| > 1$. Since $|G : N(R)| \equiv 1 \pmod 3$, $|G : N(R)| = 55$ and $|N(R)| = 3$. This is impossible when Q is not normal as a $|N(R)|$ is divisible by 5 by (b). Hence Q is normal in G. Then QR is a group of order 15 by Problem 2(b) and again R is normal in QR by the remark above, which implies that $|N(R)|$ is divisible by 5. In any case, this is a contradiction. ■