November 19, 2008

Algebra II Final 2008

In the following, when R is a commutative ring with 1, $\langle a\rangle = \{r\cdot a \mid r \in R\}$, which is denoted by (a) in the textbook.

1. Let A be a ring with 1, which may not be commutative. Suppose $xy = 0$ implies, $x = 0$ or $y = 0$. Let $a, b \in A$. Show that the following are equivalent. (10pts)
   (i) $Aa = Ab$.
   (ii) There exists $u \in U(A)$ such that $b = ua$, where u is a unit of A.

2. Let $R = \{a + b\sqrt{-1} \mid a, b \in \boldsymbol{Z}\} \subset \boldsymbol{C}$, and let $N(a + b\sqrt{-1}) = a^2 + b^2$. (40pts)
   (a) Show that R is an integral domain and $R = \{f(\sqrt{-1}) \mid f(t) \in \boldsymbol{Z}[t]\}$.
   (b) Determine the elements in $U(R)$.
   (c) Show that R is a Euclidean domain.
   (d) Determine whether each of $\langle 2\rangle$, $\langle 3\rangle$ and $\langle 5\rangle$ is a maximal ideal. If not find all the maximal ideals containing it.
   (e) Let $\langle \pi\rangle \neq \langle 0\rangle$ be a prime ideal of R. Show that there exists a prime integer p such that $\boldsymbol{Z} \cap \langle \pi\rangle = \boldsymbol{Z}\cdot p$ and that $N(\pi) = p$, or p^2. If $N(\pi) = p^2$ then $\langle p\rangle$ is a prime ideal in R and if $N(\pi) = p$ then $\langle p\rangle$ is not a prime ideal in R.

3. Let R be a commutative ring with 1. Two ideals I and J are said to be co-prime if $I + J = R$. (20pts)
   (a) Suppose I and J are co-prime ideals of R. Show that $IJ = I \cap J$ and
   $$R/IJ \simeq R/I \times R/J.$$
   (b) Suppose $I_1, I_2, \ldots, I_n$ are mutually co-prime ideals of R, i.e., I_i and I_j are co-prime if $i \neq j$. Show that for $i \in \{1, 2, \ldots, n-1\}$, $I_1 I_2 \cdots I_i$ and I_{i+1} are co-prime and
   $$\bigcap_{i=1}^{n} I_i = I_1 I_2 \cdots I_n.$$

4. Let R be a commutative ring with 1 and let $R[t]$ be the polynomial ring. Prove or disprove (by giving a counter example) the following. (20pts)
   (a) R is an integral domain if and only if $R[t]$ is an integral domain.
   (b) If R is a PID, then so is $R[t]$.
   (c) If $R[t]$ is a PID, then so is R.
   (d) If R is a Euclidean domain, then so is $R[t]$.

5. Let R be a commutative ring with 1 and let S be a multiplicative subset of R, i.e., $1 \in S$, $0 \notin S$ and $s, t \in S$ implies $st \in S$. Let I be an ideal of $S^{-1}R$. Show that $I = (\phi_S(R) \cap I)(S^{-1}R)$, where $\phi_S : R \to S^{-1}R$ $(a \mapsto a/1)$. Using this fact, show that if R is a PID, then so is $S^{-1}R$. (10pts)

Solutions to Algebra II Final 2008

In the following, when R is a commutative ring with 1, $\langle a\rangle = \{r\cdot a \mid r \in R\}$, which is denoted by (a) in the textbook.

1. Let A be a ring with 1, which may not be commutative. Suppose $xy = 0$ implies, $x = 0$ or $y = 0$. Let $a, b \in A$. Show that the following are equivalent. (10pts)

   (i) $Aa = Ab$.

   (ii) There exists $u \in U(A)$ such that $b = ua$, where u is a unit of A.

   Solution. (i)$\to$ (ii): Suppose $Aa = Ab$. Since $1 \in A$, $a \in Aa = Ab \ni b$ and there exist $c, d \in A$ such that $a = cb$ and $b = da$. Hence $a = cda$ and $b = dcb$. So $(1 - cd)a = (1 - dc)b = 0$. If $a = 0$, then $b = da$ implies $b = 0$ and similarly if $b = 0$ then $a = 0$. So if one of a or b is zero, both are zero and $a = 1b$. Hence we may assume that $a \neq 0 \neq b$. Then $(1 - cd)a = (1 - dc)b = 0$ implies $1 = cd = dc$ by hypothesis and $c, d \in U(R)$. Therefore we have (ii).

   (ii)$\to$(i). Suppose $b = ua$ and $u \in U(R)$. Then

   $$Aa = Au^{-1}b \subset Ab = Aua \subset Aa$$

   and $Aa = Ab$. ■

2. Let $R = \{a + b\sqrt{-1} \mid a, b \in \boldsymbol{Z}\} \subset \boldsymbol{C}$, and let $N(a + b\sqrt{-1}) = a^2 + b^2$. (40pts)

   (a) Show that R is an integral domain and $R = \{f(\sqrt{-1}) \mid f(t) \in \boldsymbol{Z}[t]\}$.
   Solution. Let $\phi : \boldsymbol{Z}[t] \to \boldsymbol{C}$ ($f(t) \mapsto f(\sqrt{-1})$). Then ϕ is a ring homomorphism and $\mathrm{Im}(\phi) \supset R$, as $\phi(a+bt) = a+b\sqrt{-1}$ for all $a, b \in \boldsymbol{Z}$. Suppose $f(t) \in \boldsymbol{Z}[t]$. Then there exist a polynomial $q(t) \in \boldsymbol{Z}[t]$ and $a, b \in \boldsymbol{Z}$ such that $f(t) = q(t)(t^2 + 1) + a + bt$. Since

   $$\phi(f(t)) = q(\sqrt{-1})(\sqrt{-1}^2 + 1) + a + b\sqrt{-1} = a + b\sqrt{-1} \in R,$$

   $R = \{f(\sqrt{-1}) \mid f(t) \in \boldsymbol{Z}[t]\}$. Now R is the image of a ring homomorphism ϕ, it is a subring of a field $\boldsymbol{C}$. Hence there is no zero divisor and R is an integral domain. ■

   (b) Determine the elements in $U(R)$.
   Solution. First note that for $\alpha, \beta \in R$, $N(\alpha\beta) = \alpha\beta\overline{\alpha\beta} = \alpha\overline{\alpha}\beta\overline{\beta} = N(\alpha)N(\beta)$. We claim that $U(R) = \{\pm 1, \pm\sqrt{-1}\}$. It is clear that $U(R) \supset \{\pm 1, \pm\sqrt{-1}\}$. Let $\alpha = a + b\sqrt{-1} \in U(R)$ and $\alpha\beta = 1$ for some $\beta \in R$. Then $1 = N(1) = N(\alpha\beta) = N(\alpha)N(\beta)$. Since $N(\alpha) = a^2 + b^2 \geq 0$, $a^2 + b^2 = 1$ and $(a, b) = (\pm 1, 0)$ and $(0, \pm 1)$ are the only solutions. Thus we have our claim. ■

   (c) Show that R is a Euclidean domain.
   Solution. Let $\delta(\alpha) = N(\alpha)$ for $\alpha \in R$. It is clear that if $\alpha, \beta \in R$ are nonzero, $N(\alpha), N(\beta) \geq 1$. Thus $N(\alpha\beta) \geq N(\alpha)$. Let $\alpha, \beta \in R$ with $\beta \neq 0$. Then there exist $a, b \in \boldsymbol{Q}$ such that $\alpha/\beta = a + b\sqrt{-1}$. Then we can choose $c, d \in \boldsymbol{Z}$ such that $|a - c| \leq \frac{1}{2}$ and $|b - d| \leq \frac{1}{2}$. Let $\gamma = c + d\sqrt{-1} \in R$. Then

   $$\alpha/\beta = \gamma + (a - c) + (b - d)\sqrt{-1} \text{ with } (a - c)^2 + (b - d)^2 \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1,$$

   Hence $\alpha = \beta\gamma + \beta((a - c) + (b - d)\sqrt{-1})$ with $\beta((a - c) + (b - d)\sqrt{-1}) \in R$ and $N(\beta((a - c) + (b - d)\sqrt{-1})) < N(\beta)$. Therefore R is a Euclidean domain. ■

(d) Determine whether each of $\langle 2\rangle$, $\langle 3\rangle$ and $\langle 5\rangle$ is a maximal ideal. If not find all the maximal ideals containing it.

Solution. Note that since R is a Euclidean domain, it is a principal ideal domain and a unique factorization domain. Hence $0 \neq \alpha \in R$ is an irreducible element of R if and only if $\langle \alpha \rangle$ is a maximal ideal.

Let p be a prime integer. Suppose $p = \alpha\beta$ with $\alpha, \beta \in R$. Then $p^2 = N(p) = N(\alpha)N(\beta)$. Hence if p is not irreducible and $\alpha, \beta \notin U(R)$, then $N(\alpha) = N(\beta) = p$. So if $\alpha = a + b\sqrt{-1}$, then $p = a^2 + b^2$. Clearly 3 is not expressible as a sum of two squares of integers. 3 is irreducible and $\langle 3\rangle$ is a maximal ideal.

On the other hand, $2 = (1+\sqrt{-1})(1-\sqrt{-1})$ and $5 = (1+2\sqrt{-1})(1-2\sqrt{-1})$, $\langle 1+\sqrt{-1}\rangle$ and $\langle 1-\sqrt{-1}\rangle$ are maximal ideals containing $\langle 2\rangle$ and $\langle 1+2\sqrt{-1}\rangle$ and $\langle 1-2\sqrt{-1}\rangle$ are maximal ideals containing $\langle 2\rangle$. Note that the generators of these ideals are irreducible elements as their norms, i.e., the value of N, are prime numbers. Moreover, there are no other maximal ideals containing $\langle 2\rangle$ and $\langle 5\rangle$ because if R is a principal ideal domain and hence a unique factorization domain. ■

(e) Let $\langle \pi\rangle \neq \langle 0\rangle$ be a prime ideal of R. Show that there exists a prime integer p such that $\boldsymbol{Z} \cap \langle \pi\rangle = \boldsymbol{Z} \cdot p$ and that $N(\pi) = p$, or p^2. If $N(\pi) = p^2$ then $\langle p\rangle$ is a prime ideal in R and if $N(\pi) = p$ then $\langle p\rangle$ is not a prime ideal in R.

Solution. Since R is a principal ideal domain, π is an irreducible element. Clearly $\boldsymbol{Z} \cap \langle \pi\rangle$ is a prime ideal of $\boldsymbol{Z}$. Hence there exists a prime integer p such that $\boldsymbol{Z} \cap \langle \pi\rangle = \boldsymbol{Z} \cdot p$. Since $p \in \langle \pi\rangle$, there exists $\alpha \in R$ such that $p = \alpha\pi$. Hence $p^2 = N(\alpha)N(\pi)$. If $N(\pi) = p$, then p is not irreducible and $\langle p\rangle$ is not a prime ideal in R, while if $N(\pi) = p^2$ then $\langle p\rangle$ is a prime ideal in R as p itself is irreducible. ■

3. Let R be a commutative ring with 1. Two ideals I and J are said to be co-prime if $I + J = R$. (20pts)

(a) Suppose I and J are co-prime ideals of R. Show that $IJ = I \cap J$ and

$$R/IJ \simeq R/I \times R/J.$$

Solution. Since both I and J are ideals, $IJ \subset I \cap J$. Let $x \in I \cap J$. Since $I + J = R$, there exist $u \in I$ and $v \in J$ such that $u + v = 1$. Now $x = x1 = ux + xv \in IJ$ and hence $IJ = I \cap J$.

Let $\phi : R \mapsto R/I \times R/J$ $(x \mapsto (x + I, x + J))$. Clearly the kernel of ϕ is $I \cap J$ which is equal to IJ. Hence it suffices to show that ϕ is onto. Let $(x + I, y + j) \in R/I \times R/J$. Now as $u + v = 1$ with $u \in I$ and $v \in J$,

$$\begin{aligned} \phi(uy + vx) &= (uy + vx + I, uy + vx + J) = (vx + I, uy + J) \\ &= ((1 - u)x + I, (1 - v)y + J) = (x + I, y + J). \end{aligned}$$

Therefore ϕ is onto and the above isomorphism is established. ■

(b) Suppose $I_1, I_2, \ldots, I_n$ are mutually co-prime ideals of R, i.e., I_i and I_j are co-prime if $i \neq j$. Show that for $i \in \{1, 2, \ldots, n-1\}$, $I_1 I_2 \cdots I_i$ and I_{i+1} are co-prime and

$$\bigcap_{i=1}^{n} I_i = I_1 I_2 \cdots I_n.$$

Solution. We prove by induction. If $i = 1$, there is nothing to prove by (a). Suppose the assertion holds when $i - 1 \geq 1$. Let $J = I_1 I_2 \cdots I_{i-1}$ and $J' = I_1 I_2 \cdots I_{i-2} I_i$. Then by induction hypothesis $J + I_{i+1} = R$, and $J' + I_{i+1} = R$. Therefore there exist $x \in J$, $x' \in J'$ and $y, y' \in I_{i+1}$ such that $x + y = 1$ and $x' + y' = 1$. Now $1 = (x + y)(x' + y') = xx' + xy' + x'y + yy' \in I_1 I_2 \cdots I_i + I_{i+1}$.

Now by (a) and induction the last assertion holds. ■

4. Let R be a commutative ring with 1 and let $R[t]$ be the polynomial ring. Prove or disprove (by giving a counter example) the following. (20pts)

   (a) R is an integral domain if and only if $R[t]$ is an integral domain.

   Solution. Let $0 \neq f = a_0 + a_1 t + \cdots + a_m t^m$ with $a_m \neq 0$ and $0 \neq g = b_0 + b_1 t + \cdots + b_n t^n$ with $b_n \neq 0$. Then $fg = a_0 b_0 + \cdots + a_m b_n t^{m+n}$ with $a_m b_n \neq 0$ as R is an integral domain. Hence $R[t]$ is an integral domain. ■

   (b) If R is a PID, then so is $R[t]$.

   Solution. Let $R = \boldsymbol{Z}$. Then R is a Euclidean domain and hence it is a PID. But $\boldsymbol{Z}[t]$ is not a PID as t is an irreducible element in $\boldsymbol{Z}[t]$ but $\boldsymbol{Z}[t]/\langle t \rangle \simeq \boldsymbol{Z}$ is not a field. Note that if $\boldsymbol{Z}[t]$ is a PID, the ideal generated by an irreducible element is maximal. ■

   (c) If $R[t]$ is a PID, then so is R.

   Solution. $R[t]$ is a PID if and only if R is a field. Hence R is a PID. ■

   (d) If R is a Euclidean domain, then so is $R[t]$.

   Solution. Let $R = \boldsymbol{Z}$. Then as we have seen above, $\boldsymbol{Z}[t]$ is not a PID. So it is not a Euclidean dommain. ■

5. Let R be a commutative ring with 1 and let S be a multiplicative subset of R, i.e., $1 \in S$, $0 \notin S$ and $s, t \in S$ implies $st \in S$. Let I be an ideal of $S^{-1}R$. Show that $I = (\phi_S(R) \cap I)(S^{-1}R)$, where $\phi_S : R \to S^{-1}R$ $(a \mapsto a/1)$. Using this fact, show that if R is a PID, then so is $S^{-1}R$. (10pts)

   Solution. Clearly $I \supset (\phi_S(R) \cap I)(S^{-1}R)$. Suppose $a/s \in I$. Then $a = s(a/s) \in I$. Hence $a/1 \in \phi_S(R)$. Thus $I \subset (\phi_S(R) \cap I)(S^{-1}R)$ and $I = (\phi_S(R) \cap I)(S^{-1}R)$.

   Suppose R is a PID. Then $\phi_S^{-1}(I)$ is an ideal of R and generated by an element $a \in R$. So $\langle a \rangle = \phi_S^{-1}(I)$ and $\phi_S(\langle a \rangle) = \phi_S(R) \cap I$. Hence

$$I = (\phi_S(R) \cap I)(S^{-1}R) = \langle a \rangle_{S^{-1}R}.$$

   Therefore $S^{-1}R$ is a PID. ■