BCM I : Final 2015

June 24, 2015

ID#: Name:

1. Let P, Q, R be statements.

   (a) Complete the following truth table. (4 pts)

P	Q	R	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$	$\sim R \Rightarrow (\sim P \wedge \sim Q)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

   (b) Express the contrapositive of $\sim R \Rightarrow (\sim P \wedge \sim Q)$ without using $\wedge$. (3 pts)

   (c) Express $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ with $\vee$ and $\sim$ only without using $\wedge$ nor $\Rightarrow$. (3 pts)

2. Let n be a (fixed) positive integer. For $a, b \in \boldsymbol{Z}$, we write $a \equiv b \pmod{n}$, whenever there is an integer c such that $b - a = cn$. Show the following.

   (a) Let $a, b, c, d \in \boldsymbol{Z}$. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$. (10 pts)

1. (10)	2. (20)	3. (10)	4. (20)	5. (25)	6. (15)	Total

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(b) If x is an odd integer, then $x^2 \equiv 1 \pmod 8$. (5 pts)

(c) If x, y and z are odd integers, then $x^2 + y^2 + z^2$ cannot be a square of an integer. (5 pts)

3. Show that there is an integer m such that for each integer $n \geq m$, there are <u>nonnegative</u> integers a and b such that $n = 5a + 8b$. (10 pts)

ID#: Name:

4. Let $f : X \to Y$, $g : Y \to Z$ and $h = g \circ f : X \to Z$ ($x \mapsto g(f(x))$) be functions. Prove or disprove the following.

   (a) If f is one-to-one and g is onto, then h is onto. (5 pts)

   (b) If h is one-to-one, then f is one-to-one. (5 pts)

   (c) If f is onto and h is one-to-one, then g is one-to-one. (5 pts)

   (d) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B in X. (5 pts)

ID#: Name:

5. For $a, b \in \boldsymbol{R}$ with $a < b$, let $(a, b) = \{x \in \boldsymbol{R} : a < x < b\}$, and let $\boldsymbol{R}^{>0}$ be the set of positive real numbers.

   (a) Let $f : (0,1) \to \boldsymbol{R}^{>0}$ $(x \mapsto \frac{x}{1-x})$. Show that f is one-to-one and onto. (5 pts)

   (b) State the definitions of $|A| = |B|$ and $|A| < |B|$ for subsets A, B of $\boldsymbol{R}^{>0}$. (5 pts)

(c) Let $\mathcal{P}(\boldsymbol{R}^{>0})$ be the power set of $\boldsymbol{R}^{>0}$. Then $|A| = |B|$ for $A, B \in \mathcal{P}(\boldsymbol{R}^{>0})$ is an equivalence relation on $\mathcal{P}(\boldsymbol{R}^{>0})$. (10 pts)

(d) Show that $|(a, b)| = |\boldsymbol{R}^{>0}|$ for all pairs $a, b \in \boldsymbol{R}$ with $0 \leq a < b$. (5 pts)

ID#: Name:

6. Let $S = \{\{a, b\} : a, b \in \boldsymbol{N}, a \neq b\}$ be the set of all pairs of $\boldsymbol{N}$. Let

$$g : \boldsymbol{N} \times \boldsymbol{N} \to S\ ((x, y) \mapsto \{x, x + y\}),\ \text{and}\ h : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}\ ((x, y) \mapsto 2^{x-1}3^{y-1}).$$

(a) Show that g is a one-to-one and onto function. (5 pts)

(b) Show that h is a one-to-one function. (5 pts)

(c) Show that S is denumerable. (5 pts)

Please write your comments:

(1) About this course, especially suggestions for improvements.

(2) Topics in Mathematics or in other subjects you want to study.

BCM I: Solutions to Final 2015

June 24, 2015

1. Let P, Q, R be statements.

   (a) Complete the following truth table. (4 pts)

P	Q	R	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$	$\sim R \Rightarrow (\sim P \wedge \sim Q)$
T	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
T	T	F	F $\boldsymbol{F}$ F	T $\boldsymbol{F}$
T	F	T	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	F	F $\boldsymbol{F}$	T $\boldsymbol{F}$
F	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
F	T	F	$\boldsymbol{F}$ F	T $\boldsymbol{F}$
F	F	T	$\boldsymbol{T}$	$\boldsymbol{T}$ T
F	F	F	$\boldsymbol{T}$	T $\boldsymbol{T}$ T

   (b) Express the contrapositive of $\sim R \Rightarrow (\sim P \wedge \sim Q)$ without using $\wedge$. (3 pts)

$$\sim R \Rightarrow (\sim P \wedge \sim Q) \equiv \sim (\sim P \wedge \sim Q) \Rightarrow R \equiv (P \vee Q) \Rightarrow R.$$

   (c) Express $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ with $\vee$ and $\sim$ only without using $\wedge$ nor $\Rightarrow$. (3 pts)

$$(P \Rightarrow R) \wedge (Q \Rightarrow R) \equiv \sim R \Rightarrow (\sim P \wedge \sim Q) \equiv (P \vee Q) \Rightarrow R \equiv \sim (P \vee Q) \vee R.$$

2. Let n be a (fixed) positive integer. For $a, b \in \boldsymbol{Z}$, we write $a \equiv b \pmod{n}$, whenever there is an integer c such that $b - a = cn$. Show the following.

   (a) Let $a, b, c, d \in \boldsymbol{Z}$. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$. (10 pts)

   Soln. Since $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, there exist $s, t \in \boldsymbol{Z}$ such that $b - a = sn$, $d - c = tn$. Therefore,

$$(b + d) - (a + c) = (b - a) + (d - c) = sn + tn = (s + t)n, \text{ and}$$

$$bd - ac = b(d - c) + (b - a)c = tnb + snc = (tb + sc)n.$$

   Hence $a + c \equiv b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$.

   (b) If x is an odd integer, then $x^2 \equiv 1 \pmod{8}$. (5 pts)

   Soln. Since x is odd, $x \equiv \pm 1$ or $\pm 3 \pmod{8}$. Therefore, $x^2 \equiv 1 \pmod{8}$ in all cases. ■

   (c) If x, y and z are odd integers, then $x^2 + y^2 + z^2$ cannot be a square of an integer. (5 pts)

   Soln. By way of contradiction, assume that $x^2 + y^2 + z^2 = w^2$. Then by (b), $w^2 \equiv x^2 + y^2 + z^2 \equiv 1 + 1 + 1 \equiv 3 \pmod{8}$. Hence w is odd and $w^2 \equiv 1 \pmod{8}$, which is a contradiction. Therefore, $x^2 + y^2 + z^2$ cannot be a square of an integer. ■

3. Show that there is an integer m such that for each integer $n \geq m$, there are nonnegative integers a and b such that $n = 5a + 8b$. (10 pts)

Soln. We show that $m = 28$ satisfies the condition. Note that it is impossible to express 27 in this form. $28 = 5 \cdot 4 + 8 \cdot 1$, $29 = 5 \cdot 1 + 8 \cdot 3$, $30 = 5 \cdot 6 + 8 \cdot 0$, $31 = 5 \cdot 3 + 8 \cdot 2$, $32 = 5 \cdot 0 + 8 \cdot 4$. So asume $n \geq 33$. Then $n - 5 \geq 28$ and by induction hypothesis, there are nonnegative integers a and b such that $n - 5 = 5a + 8b$. Hence $n = 5(a+1) + 8b$ and n is a nonnegative linear combination of 5 and 8. Thus by induction, for each integer $n \geq 28$, there are nonnegative integers a and b such that $n = 5a + 8b$. ■

4. Let $f : X \to Y$, $g : Y \to Z$ and $h = g \circ f : X \to Z$ $(x \mapsto g(f(x)))$ be functions. Prove or disprove the following.

(a) If f is one-to-one and g is onto, then h is onto. (5 pts)

Soln. **False.** Let $X = \{1\}$, $Y = \{1, 2\} = Z$ and $f(1) = 1$. $g(1) = 1$ and $g(2) = 2$. Then f is one-to-one and g is onto. But $h(1) = 1$ and h is not onto. ■

(b) If h is one-to-one, then f is one-to-one. (5 pts)

Soln. **True.** Suppose $f(x) = f(x')$ for $x, x' \in X$. Then $h(x) = g(f(x)) = g(f(x')) = h(x')$. Since h is one-to-one, $x = x'$. Hence f is one-to-one. ■

(c) If f is onto and h is one-to-one, then g is one-to-one. (5 pts)

Soln. **True.** Suppose $g(y) = g(y')$ for $y, y' \in Y$. Since f is onto, there exist $x, x' \in X$ such that $y = f(x)$ and $y' = f(x')$. Hence $h(x) = g(f(x)) = g(y) = g(y') = g(f(x')) = h(x')$. Since h is one-to-one, $x = x'$. Therefore, $y = f(x) = f(x') = y'$ and g is one-to-one. ■

(d) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B in X. (5 pts)

Soln. **False.** Let $X = \{1, 2\}$, $Y = \{1\}$, $f(1) = f(2) = 1$ and $A = \{1\}$, $B = \{2\}$. Then $A \cap B = \emptyset$. Hence $f(A \cap B) = f(\emptyset) = \emptyset$. On the other hand, $f(A) = f(\{1\}) = \{1\} = f(\{2\}) = f(B)$. Hence $f(A) \cap f(B) = \{1\} \neq \emptyset = f(A \cap B)$. ■

5. For $a, b \in \boldsymbol{R}$ with $a < b$, let $(a, b) = \{x \in \boldsymbol{R} : a < x < b\}$, and let $\boldsymbol{R}^{>0}$ be the set of positive real numbers.

(a) Let $f : (0, 1) \to \boldsymbol{R}^{>0}$ $(x \mapsto \frac{x}{1-x})$. Show that f is one-to-one and onto. (5 pts)

Soln. For $x \in (0, 1)$, $0 < \frac{x}{1-x}$ is well-defined and continuous as $x \neq 1$ and $f(x) \in \boldsymbol{R}^{>0}$. Moreover,

$$f'(x) = \frac{1}{(1-x)^2} > 0, \ \lim_{x \to 0} f(x) = 0 \text{ and } \lim_{x \to 1} f(x) = \infty.$$

Therefore, f is increasing and hence one-to-one. By the Intermediate Value Theorem, $f(x)$ takes all values in $\boldsymbol{R}^{>0}$. Hence f is onto. ■

Soln.2. If $f(x) = f(x')$, then $x/(1-x) = x'/(1-x')$. Hence $x(1-x') = x'(1-x)$ or $x = x'$. Thus f is one-to-one. Since $x/(1-x) = y$ implies $y = x(1+y)$ or $x = y/(1+y)$. Hence for $y \in \boldsymbol{R}^{>0}$, $x = y/(1+y) \in (0, 1)$ and $f(y/(1+y)) = y$. Therefore, f is onto. ■

(b) State the definitions of $|A| = |B|$ and $|A| < |B|$ for subsets A, B of $\boldsymbol{R}^{>0}$. (5 pts)

Soln. $|A| = |B|$ whenever $A = B = \emptyset$ or there is a one-to-one and onto function $f : A \to B$. $|A| < |B|$ whenever $A = \emptyset$ or there is a one-to-one function $g : A \to B$, and $|A| \neq |B|$. ■

(c) Let $\mathcal{P}(\boldsymbol{R}^{>0})$ be the power set of $\boldsymbol{R}^{>0}$. Then $|A| = |B|$ for $A, B \in \mathcal{P}(\boldsymbol{R}^{>0})$ is an equivalence relation on $\mathcal{P}(\boldsymbol{R}^{>0})$. (10 pts)

Soln. Let $A \in \mathcal{P}(\boldsymbol{R}^{>0})$, i.e., $A \subseteq \boldsymbol{R}^{>0}$. If $A = \emptyset$, then $|A| = |A|$. Suppose not. Then the identity function $i_A : A \to A$ $(x \mapsto x)$ is a one-to-one and onto function. Hence $|A| = |A|$. Hence the relation $|A| = |B|$ is reflexive.

Let $A, B \in \mathcal{P}(\boldsymbol{R}^{>0})$ with $|A| = |B|$. If $A = B = \emptyset$, then $|B| = |A|$. Suppose not. Then there is a one-to-one and onto function $f : A \to B$. Then $g = f^{-1} : B \to A$, the inverse function of f is also one-to-one and onto. Hence $|B| = |A|$. Hence the relation $|A| = |B|$ is symmetric.

Let $A, B, C \in \mathcal{P}(\boldsymbol{R}^{>0})$ with $|A| = |B|$ and $|B| = |C|$. If one of A, B, C is an empty set, so are all three. Hence $|A| = |C|$. Suppose not. Then there are one-to-one and onto functions $f : A \to B$ and $g : B \to C$. Then $h = g \circ f : A \to C$, the composite function of f and g is also one-to-one and onto. Hence $|A| = |C|$. Therefore the relation $|A| = |B|$ is transitive and this is an equivalence relation. ■

(d) Show that $|(a, b)| = |\boldsymbol{R}^{>0}|$ for all pairs $a, b \in \boldsymbol{R}$ with $0 \le a < b$. (5 pts)

Soln. Let $f : (0, 1) \to (a, b)$ $(x \mapsto (b - a)x + a)$. Since $b - a > 0$, f is an increasing function and $f((0, 1)) = (a, b)$. Hence f is a one-to-one and onto function. By definition, $|(0, 1)| = |(a, b)|$. Since $|A| = |B|$ is an equivalence relation and $|(0, 1)| = |\boldsymbol{R}^{>0}|$ by (a), $|(a, b)| = |\boldsymbol{R}^{>0}|$. ■

6. Let $S = \{\{a, b\} : a, b \in \boldsymbol{N}, a \neq b\}$ be the set of all pairs of $\boldsymbol{N}$. Let

$$g : \boldsymbol{N} \times \boldsymbol{N} \to S \ ((x, y) \mapsto \{x, x + y\}), \text{ and } h : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N} \ ((x, y) \mapsto 2^{x-1}3^{y-1}).$$

(a) Show that g is a one-to-one and onto function. (5 pts)

Soln. If $g(x, y) = g(x', y')$, then $\{x, x + y\} = \{x', x' + y'\}$. Since $x < x + y$ and $x' < x' + y'$, $x = x'$ and $y = y'$. Therefore g is one-to-one. Let $\{a, b\} \in S$ with $a < b$. Then $(a, b - a) \in \boldsymbol{N} \times \boldsymbol{N}$ and $g(a, b - a) = \{a, b\}$. Hence g is onto. ■

(b) Show that h is a one-to-one function. (5 pts)

Soln. Suppose $h(x, y) = h(x', y')$. Then $2^{x-1}3^{y-1} = 2^{x'-1}3^{y'-1}$. By the unique factorization of integers, $x = x'$ and $y = y'$. Hence h is one-to-one. ■

(c) Show that S is denumerable. (5 pts)

Soln. Let $f : \boldsymbol{N} \to \boldsymbol{N} \times \boldsymbol{N}$ $(x \mapsto (x, 1))$. Then f is clearly a one-to-one function. Therefore, we have

$$|\boldsymbol{N}| \le |\boldsymbol{N} \times \boldsymbol{N}| \stackrel{(a)}{=} |S|, \text{ and } |\boldsymbol{N} \times \boldsymbol{N}| \stackrel{(b)}{\le} |\boldsymbol{N}|.$$

Hence by the Cantor-Bernstein's Theorem $|\boldsymbol{N}| = |\boldsymbol{N} \times \boldsymbol{N}|$ and $|\boldsymbol{N}| = |S|$. ■

数学通論 I を受講したみなさんへ

Grading Policy

最初に配布したシラバスにあるように、演習 (Recitation) (30%) （全員に 15 問割り当てました, 4/22 にはボランティアで多くの人が 1 問余分に解いてくれました)、宿題 (Homework) (20%) （8 回、80 問提出を求めました)、期末試験 (Final) (50%) （6 月 24 日に実施)。

教員によって考え方は異なりますが、私は、授業科目というより、コースという考え方が、学士課程教育では大切だと思っています。このコースで 10 週間かけてどれだけ学んだかが重要です。学んだ内容も、学び全体の中でそれをどのように位置づけるかも、ひとそれぞれでしょう。そこで、今までの課題を丁寧に提出し、演習の問題の大部分を黒板で発表してきた人は成績 D 以上は保証します。ただし、期末試験の割合を高くしてあります。数学において、学んだ数学を試験で表現できることは大切だと考えているからです。

Final および提出物は遅くとも週明けには返却できると思います。返却できるようになった時点で Moodle 出知らせ、授業支援室 H113（H113 閉室のときは H109）で受け取るようにします。

専門の数学の最初のコースはどうでしたか。楽しめましたか。お疲れ様。

After BCM I

まずは、夏の数学セミナーへ参加して下さると嬉しいですね。(8 月 24 日から 8 月 27 日の 4 日間です。場所は ICU 軽井沢キャンパスです。）今年の教科書はもうすぐ決まると思います。私のホームページに過去の記録が載っています。

http://subsite.icu.ac.jp/people/hsuzuki/science/class/summer_seminar/index.html

また、教科書の第 13 章 Proofs in Calculus を読むことを勧めます。微分積分学を復習することにもなりますし、数学通論 II への橋渡しにもなります。数学通論に関連して、私が学生時代に読んだ本を中心にお薦めを書きます。他にも読みやすい良い本が最近はたくさん出ています。

1. 数学通論 III の参考書「集合と位相」内田伏一著、裳華房 の前半が集合論です。
2. 数学通論 II に関係する、高木貞治「解析概論」を最初からじっくり読む。
3. 線形代数学 III に関係する、佐武一郎著「線型代数入門」を最初からじっくり読む。
4. 簡単に取り組めそうなのは、「集合への 30 講」志賀浩二著　朝倉書店。
5. 公理的集合論入門をかじってみたい人には、「新装版：集合とはなにか (はじめて学ぶ人のために)」竹内外史著、講談社。

私は、一年生の夏休みは岩村聯著「束論」を読み通しました。これが数学の本で初めて読み通したものでした。

二年生の夏は記憶が定かではありませんが松坂和夫著「集合と位相」を読んだと思います。全部は読まなかったかも知れません。個人的には、上の 1 にかわるものとしてお薦めです。

三年生の夏には「集合論入門」赤摂也著、培風館（ISBN4-563-00301-8, 1957.1.25）を短期間に読み通しました。集合と位相関係の本が並びましたが、無論、他にも読みました。Serge Lang の Algebra は一年生の秋から、3 人で自主ゼミをして読み、4 年まで続けました。完全には終わりませんでした。ポントリャーギンの「連続群論」上下も 3 人での自主ゼミをながいことしましたが、上巻しかゼミでは終わりませんでした。ポントリャーギンの「常微分方程式」はかなり進みましたが、読み終わったかどうかはあまりよく覚えていません。コルモゴロフ・フォミーンの「関数解析の基礎」は読み始めましたが、問題が難しく、あまり進みませんでした。

夏休みにじっくり一冊、数学の本を読むことに時間をかけることができれば、たとえ、読む量は少なくても、大きな価値があると思いますよ。数学で学んだことは何年かたって、誤りだったということも、時代遅れになることもありません。また、苦労して読む経験はすべて思考のトレーニングになっているハズです。s 少しずつ理解し始めると数学は本当に楽しいですよ。

鈴木寛 (*hsuzuki@icu.ac.jp*, URL: http://subsite.icu.ac.jp/people/hsuzuki/science/)