Introduction to Linear Algebra *November 18, 2010*

Final Exam 2010

(Total: 100 pts)

Division: **ID#:** **Name:**

1. Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be as follows. (10 pts)

$$\boldsymbol{u} = (4, -8, 1),\ \boldsymbol{v} = (2, 1, -2),\ \boldsymbol{w} = (3, -4, 12).$$

(a) The vector $\boldsymbol{p} = \mathrm{proj}_{\boldsymbol{v}}\boldsymbol{u}$ is a scalar multiple of $\boldsymbol{v}$ such that $\boldsymbol{u} - \boldsymbol{p}$ is orthogonal to $\boldsymbol{v}$. Find $\boldsymbol{p}$. (Show work.)

(b) Compute $\boldsymbol{u} \times \boldsymbol{v}$, and find the volume of the parallelepiped (*heiko-6-mentai*) in 3-space determined by the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. (Show work.)

Points:

1.(*a*)	(*b*)	2.∗	3.(*a*)∗	(*b*)	(*c*)	(*d*)	4.(*a*)	(*b*)	**Total**
(*c*)	(*d*)	(*e*)	5(*a*)	(*b*)	(*c*)	(*d*)	(*e*)∗	* 10pts else 5 pts	

2. Evaluate the following determinant. Write explanation in words in detail at each step. (10 pts)

$$\begin{vmatrix} \lambda - c_1 & -c_2 & \cdots & -c_n \\ -c_1 & \lambda - c_2 & \cdots & -c_n \\ \vdots & \vdots & & \vdots \\ -c_1 & -c_2 & \cdots & \lambda - c_n \end{vmatrix} =$$

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be the matrices below. Assume $A\boldsymbol{x} = \boldsymbol{b}$.

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 1 & 4 & 12 \\ 2 & -2 & 1 & 1 \\ 2 & 1 & 1 & -3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Briefly explain each step. (10 pts)

(b) Applying the Cramer's rule and express x_1 and x_4 as quotients of determinants. Do not evaluate determinants. (5 pts)

(c) Explain that the following system of linear equations with unknowns $y_1, y_2, y_3, y_4, y_5, y_6$ is always consistent and the solution can be written with two free parameters for any a, b, c, d, e, f, g and h. (5 pts)

$$\begin{cases} y_1 & & + 2y_3 & + 3y_4 & + ay_5 & + ey_6 & = 1 \\ -2y_1 & + y_2 & + 4y_3 & + 12y_4 & + by_5 & + fy_6 & = 2 \\ 2y_1 & - 2y_2 & + y_3 & + y_4 & + cy_5 & + gy_6 & = -2 \\ 2y_1 & + y_2 & + y_3 & - 3y_4 & + dy_5 & + hy_6 & = 3 \end{cases}$$

(d) Let H and H' be as below. Suppose $A^{-1}H = H'$. Explain that the solutions to the system of linear equations in (c) can be expressed as follows, where s and t are free parameters. (5 pts)

$$H = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}, \quad H' = \begin{bmatrix} a' & e' \\ b' & f' \\ c' & g' \\ d' & h' \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 0 \\ 0 \end{bmatrix} - s \cdot \begin{bmatrix} a' \\ b' \\ c' \\ d' \\ -1 \\ 0 \end{bmatrix} - t \cdot \begin{bmatrix} e' \\ f' \\ g' \\ h' \\ 0 \\ -1 \end{bmatrix}.$$

4. Let B be the augmented matrix of a system of linear equations. Let C be a matrix obtained from B after a series of elementary row operation. (25 pts)

$$B = \begin{bmatrix} 1 & 0 & 2 & 3 & a \\ -2 & 1 & 4 & 12 & b \\ 2 & -2 & 1 & 1 & c \\ 2 & 1 & 1 & -3 & d \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 2 & 3 & a' \\ 0 & 1 & -3 & -9 & b' \\ 0 & -2 & -3 & -5 & c' \\ 0 & 1 & 8 & 18 & d' \end{bmatrix}.$$

(a) Express a', b', c' and d' in terms of a, b, c, d. (Show work.)

(b) Write the sequence of operations applied to B to obtain C using $[i;c]$, $[i,j]$, $[i,j;c]$ notation.

(c) Let P be a 4×4 matrix such that $PB = C$. Express each of P and P^{-1} as a product of elementary matrices using the notation $P(i;c)$, $P(i,j)$, $P(i,j;c)$.

(d) Determine P and P^{-1}. (Solution only.)

(e) Explain that P in (c) is uniquely determined.

5. Let A, $\boldsymbol{x}$, $\boldsymbol{b}_n$ $(n = 0, 1, 2, \ldots)$ be as follows. (30 pts)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix}, \ \boldsymbol{b}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{bmatrix} = \boldsymbol{b}_{n+1} = A\boldsymbol{b}_n = A \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}.$$

(a) Find the cofactor matrix $\tilde{A}$, the adjoint matrix $\text{adj}(A)$ and the inverse of A. (Solution only.)

(b) Find the characteristic polynomial and the eigenvalues of A. (Show work.)

(c) Find an eigenvector corresponding to each of the eigenvalues of A. (Show work.)

(d) Find a 3×3 matrix P and a diagonal matrix D such that $AP = PD$. (Give explanation.)

(e) When $a_0 = 1$, $a_1 = -4$ and $a_2 = -4$, find a_n. (Show work.)

メッセージ欄：この授業について、特に改善点について、その他何でもどうぞ。