Linear Algebra I *November 15, 2012*

Final Exam 2012

(Total: 100 pts, 40% of the grade)

ID#: **Name:**

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation defined by: (30 pts)

$$T(x_1, x_2, x_3) = (3x_1 + x_2, 2x_1 + 2x_2 - 3x_3, -3x_1 + x_2 - 5x_3).$$

   (a) Show that T is a linear transformation.

   (b) Find the standard matrix $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ for the linear transformation T.

   (c) Find $\boldsymbol{v}_1 \times \boldsymbol{v}_2$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are in (b).

Points:

1.(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	2.(*a*)$*$	(*b*)	(*c*)	*none*	**Total**
									5	
3.(*a*)	(*b*)	(*c*)	(*d*)	4.(*a*)	(*b*)	(*c*)	(*d*)	(*e*)$*$	$*$	
									10	

1. Continued from page 1.

   (d) Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \mathbb{R}^3$. Suppose the volume of the parallelepiped determined by $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is 5. What is the volume of the parallelepiped determined by $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), T(\boldsymbol{u}_3)$. Write a brief explanation.

   (e) (i) Show that there is a linear transformation $U : \mathbb{R}^3 \to \mathbb{R}^3$ $(\boldsymbol{x} = (x_1, x_2, x_3) \mapsto U(\boldsymbol{x}) = U(x_1, x_2, x_3)$ such that $U(T(x_1, x_2, x_3)) = (x_1, x_2, x_3)$, i.e., $U(T(\boldsymbol{x})) = \boldsymbol{x}$ and that (ii) the standard matrix of U is A^{-1}.

   (f) Find the $(2, 3)$ entry of A^{-1}.

2. Let A and P be the following 4×4 matrices, and $\boldsymbol{b} \in \mathbb{R}^4$ given below. (20 pts)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d & -c & -b & -a \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & 3 \\ 1 & 4 & 4 & 9 \\ 1 & 8 & -8 & 27 \end{bmatrix}.$$

(a) Find the characteristic polynomial $p(x) = \det(A - xI)$ of A.

(b) Show that if λ is an eigenvalue of A, then $\boldsymbol{b}$ is an eigenvector of A corresponding to λ.

(c) Suppose $AP = PD$ for some diagonal matrix D. Determine a, b, c, d and D.

3. Let A, B, $\boldsymbol{x}$ and $\boldsymbol{b}$ be matrices and vectors given below. Assume $A\boldsymbol{x} = \boldsymbol{b}$. (20 pts)

$$A = \begin{bmatrix} 0 & 2 & 1 & 3 & 4 \\ -2 & 2 & -3 & -2 & 2 \\ 0 & -2 & -4 & 3 & 1 \\ -3 & 3 & 1 & -7 & -2 \\ 1 & -1 & 2 & 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & -2 & -4 & 3 & 1 \\ 0 & 0 & 7 & 2 & -2 \\ 0 & 2 & 1 & 3 & 4 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \end{bmatrix}.$$

(a) The matrix B is obtained from the matrix A by applying a sequence of elementary row operations. (i) Find a matrix P such that $PA = B$, and (ii) express P as a product of elementary matrixes $E(i;c)$, $E(i,j)$, $E(i,j;c)$.

(b) Evaluate $\det(A)$. Briefly explain each step.

(c) The matrix P in (a) is uniquely determined. Give your reason.

(d) Applying the Cramer's rule and express x_2 and x_5 as quotients of determinants. Do not evaluate determinants.

4. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 12 & 6 & 4 \\ 0 & 5 & 8 \end{bmatrix}$. (30 pts)

(a) Show that the characteristic polynomial of A is equal to the characteristic polynomial of A^T.

(b) Show that 12 is an eigenvalue of A.

(c) Find an eigenvector of A corresponding to an eigenvalue 12.

(d) Find all eigenvalues of A.

(e) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

メッセージ欄：この授業について、特に改善点について、その他何でもどうぞ。