Linear Algebra I *November 21, 2013*

Final Exam 2013

(Total: 100 pts, 40% of the grade)

ID#: **Name:**

1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a transformation defined by: (30 pts)

$$T(x_1, x_2, x_3, x_4) = (3x_1 + x_2 - x_4, x_1 + 2x_2 - 3x_3 + 3x_4, -2x_1 + 4x_2 - 2x_3 + 5x_4).$$

   (a) Show that T is a linear transformation.

   (b) Find the standard matrix $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4]$ for the linear transformation T.

Points:

1.(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	2.(*a*)∗	(*b*)	(*c*)	(*d*)	**Total**
3.(*a*)∗	(*b*)∗	(*c*)	4.(*a*)	(*b*)	(*c*)∗			*none*	∗	
								5	10	

1. Continued from page 1.

   (c) Find $\boldsymbol{v}_1 \times \boldsymbol{v}_2$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are in (b).

   (d) Find the volume of the parallelepiped determined by $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, where $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are in (b).

   (e) Determine whether T is one-to-one. Explain your answer.

   (f) Determine whether T is onto. Explain your answer.

2. Let A be the following 4×4 matrix and a, b, c, d real numbers. (25 pts)

$$A = \begin{bmatrix} 1 & x_1 & {x_1}^2 & {x_1}^3 \\ 1 & x_2 & {x_2}^2 & {x_2}^3 \\ 1 & x_3 & {x_3}^2 & {x_3}^3 \\ 1 & x_4 & {x_4}^2 & {x_4}^3 \end{bmatrix}.$$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ is called a cubic polynomial.

(a) Show that $\det(A) = (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)(x_4 - x_3)$.

2. Continued from page 3.

 (b) Explain that a cubic polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ is uniquely determined when $f(1) = 2, f(2) = 0, f(3) = 1, f(4) = 3$.

 (c) Find a_3 in (b) by Cramer's rule. Don't evaluate determinants.

 (d) Suppose x_1, x_2, x_3, x_4 are distinct. Explain that a cubic polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ is uniquely determined when $f(x_1) = y_1, f(x_2) = y_2, f(x_3) = y_3, f(x_4) = y_4$ for any y_1, y_2, y_3, y_4.

3. Let A and B be matrices given below. (25 pts)

$$A = \begin{bmatrix} 3 & -5 & -5 & -4 & -2 \\ -3 & 4 & 2 & 6 & 6 \\ -3 & 3 & 0 & 6 & 9 \\ -3 & 1 & -4 & 7 & 8 \\ -3 & 6 & 6 & 6 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 0 & -2 & -3 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & -2 & -5 & 2 & 7 \\ 0 & -2 & -4 & 1 & -1 \\ 0 & 3 & 6 & 0 & -2 \end{bmatrix}.$$

(a) The matrix B is obtained from the matrix A by applying a sequence of elementary row operations. Find (i) such a sequence of elementary row operations, (ii) a matrix P such that $PA = B$, and (iii) $\det(P)$.

(b) Evaluate $\det(A)$. Briefly explain each step.

(c) Write the $(2, 4)$ entry of $\mathrm{adj}(A)$, the adjugate of A, as a determinant. Don't evaluate it.

4. Let $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ -1 & 2 & 4 & 1 \\ 1 & -2 & 2 & 5 \end{bmatrix}$. (20 pts)

(a) Explain that A has eigenvalues 6 and 0 without computing the characteristic polynomial of A.

(b) Find all eigenvalues of A.

(c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

メッセージ欄：この授業について、特に改善点について、その他何でもどうぞ。