**Linear Algebra I** *November 20, 2014*

# Final Exam 2014

(Total: 100 pts, 50% of the grade)

**ID#:** **Name:**

1. Let $\boldsymbol{u} = [2, 1, -3]^T$, $\boldsymbol{v} = [0, 1, 2]^T$, $\boldsymbol{w} = [1, 3, 1]^T$, $\boldsymbol{e}_1 = [1, 0, 0]^T$, $\boldsymbol{e}_2 = [0, 1, 0]^T$ and $\boldsymbol{e}_3 = [0, 0, 1]^T$. (10 pts)

   (a) Find $\boldsymbol{u} \times \boldsymbol{v}$ and the volume of the parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Show work!

   (b) Find the standard matrix $A$ of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\boldsymbol{e}_1) = \boldsymbol{u}$, $T(\boldsymbol{e}_1 + \boldsymbol{e}_2) = \boldsymbol{v}$ and $T(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3) = \boldsymbol{w}$. Show work!

**Points:**

| 1.(*a*) | (*b*) | 2.(*a*) | (*b*) | (*c*) | (*d*) | (*e*) | (*f*) | 3.(*a*)∗ | (*b*)∗ | **Total** |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | |
| 4.(*a*)∗ | (*b*) | (*c*) | 5.(*a*) | (*b*) | (*c*)∗ | | | *none* | ∗ | |
| | | | | | | | | 5 | 10 | |

メッセージ欄：この授業について、特に改善点について、その他何でもどうぞ。

2. Consider the system of linear equations with augmented matrix $C = [\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_5, \boldsymbol{c}_6]$, where $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_6$ are the columns of $C$. We obtained a row echelon form $G$ after applying a sequence of elementary row operations to the matrix $C$. (30 pts)

$$C = \begin{bmatrix} 0 & 0 & 1 & -2 & 0 & -7 \\ 1 & 1 & 0 & 2 & 0 & 9 \\ -1 & -1 & 0 & -1 & -1 & -6 \\ -3 & -3 & -2 & -2 & 0 & -13 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 9 \\ 0 & 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Describe each step of a sequence of elementary row operations to obtain $G$ from $C$ by $[i, j], [i, j; c], [i; c]$ notation. Show work.

(b) Find an invertible matrix $P$ of size 4 such that $G = PC$ and express $P$ as a product of elementary matrices. Show work.

(c) Is $P$ in (b) uniquely determined? Give a brief explanation.

(d) Find three columns of $C$ that are linearly independent, and find three columns of $C$ that are linearly dependent. Give a brief explanation.

(e) By applying a sequence of elementary row operations, reduce $C$ to the reduced row echelon form. Show work!

(f) Find all solutions of the system of linear equations.

3. Let $A$, $\boldsymbol{x}$ and $\boldsymbol{b}$ be a matrix and vectors given below. (20 pts)

$$A = \begin{bmatrix} 4 & -1 & 2 & 0 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 1 & 1 \\ -2 & 3 & 1 & 2 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Show work!

(b) Express $y$ as a quotient (*bun-su*) of determinants when $A\boldsymbol{x} = \boldsymbol{b}$, and write $\operatorname{adj}(A)$, the adjugate of $A$. Don't evaluate the determinants.

$y =$ , $\operatorname{adj}(A) =$

4. Let $A$ be the $6 \times 6$ matrix given below, where $a$ and $b$ are real numbers. (20 pts)

$$A = \begin{bmatrix} a & b & b & b & b & b \\ b & a & b & b & b & b \\ b & b & a & b & b & b \\ b & b & b & a & b & b \\ b & b & b & b & a & b \\ b & b & b & b & b & a \end{bmatrix}.$$

(a) Find the determinant of $A$. Show work!

(b) Find the characteristic polynomial of $A$. Give a brief explanation.

(c) Find the condition on $a$ and $b$ that the matrix linear transformation $T : \mathbb{R}^6 \to \mathbb{R}^6$ ($\boldsymbol{x} \mapsto A\boldsymbol{x}$) is onto. Give a brief explanation.

5. Let $A$ be the following matrix. (20 pts)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

(a) List all eigenvalues of $A$, and give a reason that $A$ is diagonalizable.

(b) Find an eigenvector of the largest eigenvalue of $A$. Show work!

(c) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$. Show work!