Linear Algebra I *October 11, 2012*

Midterm Exam 2012

(Total: 100 pts, 20% of the grade)

ID#: **Name:**

I. For 1 to 6, let

$$A = \begin{bmatrix} 2 & 4 & 5 & 3 \\ 1 & 3 & 0 & -1 \\ 4 & 9 & 6 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -2 \end{bmatrix}, B_0 = \begin{bmatrix} 2 & 4 & 5 & 3 & c_1 \\ 1 & 3 & 0 & -1 & c_2 \\ 4 & 9 & 6 & -1 & c_3 \\ 0 & -1 & 2 & 1 & c_4 \end{bmatrix}$$

$$B_0 \to B_1 = \begin{bmatrix} 1 & 3 & 0 & -1 & c_2 \\ 2 & 4 & 5 & 3 & c_1 \\ 4 & 9 & 6 & -1 & c_3 \\ 0 & -1 & 2 & 1 & c_4 \end{bmatrix} \longrightarrow B_2 = \begin{bmatrix} 1 & 3 & 0 & -1 & c_1' \\ 0 & -2 & 5 & 5 & c_2' \\ 0 & -3 & 6 & 3 & c_3' \\ 0 & -1 & 2 & 1 & c_4' \end{bmatrix}.$$

Matrix B_1 is obtained from B_0 by an elementary row operation and B_2 is obtained from B_1 by a sequence of elementary row operations.

1. Express c_1', c_2', c_3', c_4' in terms of c_1, c_2, c_3, c_4. (10 pts)

2. Write a sequence of elementary row operations applied to B_0 to obtain B_2 using $[i; c], [i, j], [i, j; c]$ notation, by assuming that B_1 is obtained by the first operation. (10 pts)

3. Find a 4×4 matrix P such that $PB_0 = B_2$. (10 pts)

Points:

I-1	2	3	4*	5	6	II-1*	2	**Total**

*: 20 points

4. Find a reduced row echelon form of $[A\ \boldsymbol{b}]$ and find the solution $\boldsymbol{x}$ of a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$. Show work! (20 pts)

5. Let $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4]$. Show that the columns of A form a linearly dependent set. (10 pts)

6. Find a vector in $\mathbb{R}^4$ which is not in the span of the set of column vectors of A, i.e., a vector $\boldsymbol{x}$ not in $\text{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$. (10 pts)

II. For 1 and 2 below, let

$$C = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -2 & 1 \\ -2 & 2 & -5 \end{bmatrix}. \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \boldsymbol{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, [C\ I] = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -2 & 2 & -5 & 0 & 0 & 1 \end{bmatrix}$$

1. Find the inverse of C by applying a sequence of elementary row operations to $[C\ I]$. Show work!. (20 pts)

2. Using the inverse of C obtained in the previous problem, find a solution to $C\boldsymbol{x} = \boldsymbol{d}$. Explain that $C\boldsymbol{x} = \boldsymbol{d}$ has exactly one solution. (10 pts)

Message Column: Your comments are welcome, about this course, especially for improvements. この授業について、特に改善点について、その他何でもどうぞ。

Linear Algebra I *October 11, 2012*

Solutions to Midterm Exam 2012

I. For 1 to 6, let

$$A = \begin{bmatrix} 2 & 4 & 5 & 3 \\ 1 & 3 & 0 & -1 \\ 4 & 9 & 6 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -2 \end{bmatrix}, B_0 = \begin{bmatrix} 2 & 4 & 5 & 3 & c_1 \\ 1 & 3 & 0 & -1 & c_2 \\ 4 & 9 & 6 & -1 & c_3 \\ 0 & -1 & 2 & 1 & c_4 \end{bmatrix}$$

$$B_0 \to B_1 = \begin{bmatrix} 1 & 3 & 0 & -1 & c_2 \\ 2 & 4 & 5 & 3 & c_1 \\ 4 & 9 & 6 & -1 & c_3 \\ 0 & -1 & 2 & 1 & c_4 \end{bmatrix} \longrightarrow B_2 = \begin{bmatrix} 1 & 3 & 0 & -1 & c_1' \\ 0 & -2 & 5 & 5 & c_2' \\ 0 & -3 & 6 & 3 & c_3' \\ 0 & -1 & 2 & 1 & c_4' \end{bmatrix}.$$

Matrix B_1 is obtained from B_0 by an elementary row operation and B_2 is obtained from B_1 by a sequence of elementary row operations.

1. Express c_1', c_2', c_3', c_4' in terms of c_1, c_2, c_3, c_4. (10 pts)

   Solution. $c_1' = c_2, c_2' = c_1 - 2c_2, c_3' = c_3 - 4c_2, c_4' = c_4$.

2. Write a sequence of elementary row operations applied to B_0 to obtain B_2 using $[i; c], [i, j], [i, j; c]$ notation, by assuming that B_1 is obtained by the first operation. (10 pts)

   Solution. $[1, 2] \to [2, 1; -2] \to [3, 1; -4]$

3. Find a 4×4 matrix P such that $PB_0 = B_2$. (10 pts)

   Solution.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Find a reduced row echelon form of $[A\ \boldsymbol{b}]$ and find the solution $\boldsymbol{x}$ of a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$. Show work! (20 pts)

   Solution. Since $c_1' = c_2 = 2$, $c_2' = c_1 - 2c_2 = -3$, $c_3' = -6$, $c_4' = -2$,

$$[A\ \boldsymbol{b}] \to \begin{bmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 5 & 5 & -3 \\ 0 & -3 & 6 & 3 & -6 \\ 0 & -1 & 2 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 & 2 & -4 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -16 & -10 \\ 0 & 1 & 0 & 5 & 4 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 16 \\ -5 \\ -3 \\ 1 \end{bmatrix}, \ s \text{ is a free parameter}$$

5. Let $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4]$. Show that the columns of A form a linearly dependent set. (10 pts)

Solution. Since $x_1 = 16, x_2 = -5, x_3 = -3, x_4 = 1$ is a solution to $A\boldsymbol{x} = \boldsymbol{0}$.

$$\boldsymbol{0} = A\boldsymbol{x} = 16\boldsymbol{a}_1 - 5\boldsymbol{a}_2 - 3\boldsymbol{a}_3 + \boldsymbol{a}_4 = 16\begin{bmatrix} 2 \\ 1 \\ 4 \\ 0 \end{bmatrix} - 5\begin{bmatrix} 4 \\ 3 \\ 9 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Hence the columns of A form a linearly dependent set.

6. Find a vector in $\mathbb{R}^4$ which is not in the span of the set of column vectors of A, i.e., a vector $\boldsymbol{x}$ not in Span$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$. (10 pts)

Solution. Using B_2, we obtain an echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & -1 & c_2 \\ 0 & -2 & 5 & 5 & c_1 - 2c_2 \\ 0 & -3 & 6 & 3 & c_3 - 4c_2 \\ 0 & -1 & 2 & 1 & c_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & -1 & c_2 \\ 0 & 0 & 1 & 3 & c_1 - 2c_2 - 2c_4 \\ 0 & 0 & 0 & 0 & c_3 - 4c_2 - 3c_4 \\ 0 & -1 & 2 & 1 & c_4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & -1 & c_2 \\ 0 & -1 & 2 & 1 & c_4 \\ 0 & 0 & 1 & 3 & c_1 - 2c_2 - 2c_4 \\ 0 & 0 & 0 & 0 & c_3 - 4c_2 - 3c_4 \end{bmatrix}. \text{ Let } \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Since $A\boldsymbol{x} = \boldsymbol{c}$ has a solution if and only if $c_3 - 4c_2 - 3c_4 = 0$. Since $\boldsymbol{v}$ with $c_1 = 0, c_2 = 1, c_3 = 0, c_4 = 0$ does not satisfy this condition, $A\boldsymbol{x} = \boldsymbol{v}$ does not have a solution and $\boldsymbol{v}$ cannot be expressed as a linear combination of the columns of A. Hence $\boldsymbol{v}$ is not in the span of the set of column vectors of A.

II. For 1 and 2 below, let

$$C = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -2 & 1 \\ -2 & 2 & -5 \end{bmatrix}. \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \boldsymbol{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, [C\, I] = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -2 & 2 & -5 & 0 & 0 & 1 \end{bmatrix}$$

1. Find the inverse of C by applying a sequence of elementary row operations to $[C\, I]$. Show work!. (20 pts)

Solution.

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ -2 & 2 & -5 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -3 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & -5 & -3 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & -5 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -8 & 1 & -3 \\ 0 & 1 & 0 & -13 & 1 & -5 \\ 0 & 0 & 1 & -2 & 0 & -1 \end{bmatrix}, C^{-1} = \begin{bmatrix} -8 & 1 & -3 \\ -13 & 1 & -5 \\ -2 & 0 & -1 \end{bmatrix}.$$

2. Using the inverse of C obtained in the previous problem, find a solution to $C\boldsymbol{x} = \boldsymbol{d}$. Explain that $C\boldsymbol{x} = \boldsymbol{d}$ has exactly one solution. (10 pts)

Solution.

$$\boldsymbol{x} = C^{-1}C\boldsymbol{x} = C^{-1}\boldsymbol{d} = \begin{bmatrix} -8 & 1 & -3 \\ -13 & 1 & -5 \\ -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -15 \\ -26 \\ -5 \end{bmatrix}$$

If $C\boldsymbol{x} = \boldsymbol{d}$, then by multiplying C^{-1} from the left, we have $\boldsymbol{x} = C^{-1}\boldsymbol{d}$, which is the vector above. Therefore the solution is unique.