Solutions to Take-Home Quiz 4 (*October 5, 2007*)

(This quiz is designed to give you hints to read an article titled "The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof," handed out at the second lecture.)

1. Express, if possible, the matrix below as a product of elementary matrices, if not, explain the reason. (If you apply a theorem, clarify which part is used.)

   Sol.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 3 & 9 \end{bmatrix} \xrightarrow{[2,1;-2]} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \\ 3 & 3 & 9 \end{bmatrix} \xrightarrow{[3,1;-3]} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{[2;-1]} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix}$$

$$\xrightarrow{[1,2;-2]} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{[3,2;3]} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(Reduced row echelon form)}$$

   Now we apply Theorem 5.1 (1.5.3 or 1.6.4 in the textbook). (c) $\Leftrightarrow$ (d). Since the reduced row echelon form is not the identity matrix, A is not expressible as a product of elementary matrices. (Strictly speaking (d) $\Rightarrow$ (c), or its contraposition ((d) $\Rightarrow$ (c) の対偶) is used.)

   Another solution: Since $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 3 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, by Theorem 5.1 (1.5.3 or 1.6.4 in the textbook). (b) $\Leftrightarrow$ (d), the matrix cannot be expressed as a product of elementary matrices. (Strictly speaking (d) $\Rightarrow$ (b), or its contraposition is used. Can you tell how the nontrivial solution $[-2, -1, 1]^T$ is found? It is read from the reduced row echelon form obtained above.) ■

2. We want to show "the reduced row echelon form of a matrix is unique." Let A be an $m \times n$ matrix and let both B and C be reduced row echelon form of A. Since B and C are obtained by performing a series to elementary row operations to A, there are invertible matrices P and Q such that $B = PA$ and $C = QA$.

   (a) Let $\boldsymbol{x}$ be an $n \times 1$ matrix. Show that $A\boldsymbol{x} = \mathbf{0} \Leftrightarrow B\boldsymbol{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix of size $n \times 1$.

   Sol. Suppose $A\boldsymbol{x} = \mathbf{0}$. Then $B\boldsymbol{x} = PA\boldsymbol{x} = P\mathbf{0} = \mathbf{0}$. Conversely suppose $B\boldsymbol{x} = \mathbf{0}$. Since P is invertible, $A\boldsymbol{x} = P^{-1}PA\boldsymbol{x} = P^{-1}B\boldsymbol{x} = P^{-1}\mathbf{0} = \mathbf{0}$. ■

   (b) Let $\boldsymbol{x}$ be an $n \times 1$ matrix. Show that if $A\boldsymbol{x} = \mathbf{0}$, then $(B - C)\boldsymbol{x} = \mathbf{0}$.

   Sol. By (a) we have $B\boldsymbol{x} = \mathbf{0} \Leftrightarrow A\boldsymbol{x} = \mathbf{0} \Leftrightarrow C\boldsymbol{x} = \mathbf{0}$, as $C = QA$ and Q is invertible. Suppose $A\boldsymbol{x} = \mathbf{0}$. Then $B\boldsymbol{x} = \mathbf{0} = C\boldsymbol{x}$. Hence $(B - C)\boldsymbol{x} = B\boldsymbol{x} - C\boldsymbol{x} = \mathbf{0} - \mathbf{0} = \mathbf{0}$. ■

Please read the article and understand its proof. It may be a little difficult but you can understand.