Solutions to Take-Home Quiz 6 (*October 19, 2007*)

1. In the following we consider the equation $A\boldsymbol{x} = \boldsymbol{b}$.

   (a) Evaluate $\det(A)$, and determine whether there is no solution, exactly one solution or infinitely many solutions.

   Sol.

$$\begin{vmatrix} 2 & -2 & -4 & 0 \\ -3 & 5 & 4 & 5 \\ 4 & 2 & -5 & 3 \\ 5 & -7 & -3 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & -2 & 0 \\ -3 & 5 & 4 & 5 \\ 4 & 2 & -5 & 3 \\ 5 & -7 & -3 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & -2 & 0 \\ 0 & 2 & -2 & 5 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & 7 & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & -2 & 5 \\ 6 & 3 & 3 \\ -2 & 7 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & -2 & 5 \\ 0 & 9 & -12 \\ 0 & 5 & 5 \end{vmatrix}$$

$$= 2 \cdot 2 \cdot 3 \cdot 5 \cdot \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 = 420.$$

   (b) By Cramer's rule express $x_3 = \frac{\det(B)}{\det(A)}$ as a fraction of two determinants. Write down the matrix B in the numerator.

   Sol.

$$B = \begin{bmatrix} 2 & -2 & 3 & 0 \\ -3 & 5 & -2 & 5 \\ 4 & 2 & 1 & 3 \\ 5 & -7 & 0 & 0 \end{bmatrix}$$

   (c) Evaluate $\det(B)$ in the previous problem and find x_3.

   Sol.

$$|B| = \begin{vmatrix} 2 & -2 & 3 & 0 \\ -3 & 5 & -2 & 5 \\ 4 & 2 & 1 & 3 \\ 5 & -7 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -10 & -8 & 3 & -9 \\ 5 & 9 & -2 & 11 \\ 0 & 0 & 1 & 0 \\ 5 & -7 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -10 & -8 & -9 \\ 5 & 9 & 11 \\ 5 & -7 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 10 & 13 \\ 5 & 9 & 11 \\ 0 & -16 & -11 \end{vmatrix} = -5\begin{vmatrix} 10 & 13 \\ -16 & -11 \end{vmatrix} = -5(10 \cdot (-11) - 13 \cdot (-16))$$

$$= -490, \quad x_3 = \frac{-490}{420} = -\frac{7}{6}.$$

2. Evaluate the determinant of T.

   Sol.

$$|T| = \begin{vmatrix} a+b+2c & a+b+2c & a+b+2c & a+b+2c \\ b & a & c & c \\ c & c & a & b \\ c & c & b & a \end{vmatrix} = (a+b+2c)\begin{vmatrix} 1 & 1 & 1 & 1 \\ b & a & c & c \\ c & c & a & b \\ c & c & b & a \end{vmatrix}$$

$$= (a+b+2c)\begin{vmatrix} 1 & 0 & 0 & 0 \\ b & a-b & c-b & c-b \\ c & 0 & a-c & b-c \\ c & 0 & b-c & a-c \end{vmatrix} = (a+b+2c)(a-b)((a-c)^2 - (b-c)^2)$$

$$= (a-b)^2(a+b+2c)(a+b-2c) = a^4 - 2a^2b^2 + 8abc^2 - 4c^2a^2 + b^4 - 4b^2c^2.$$