Solutions to Take-Home Quiz 3 (*October 1, 2010*)

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be as follows.

$$A = \begin{bmatrix} -2 & 1 & 4 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & -3 & 1 & -1 \\ 1 & 3 & -2 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{c} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

1. Find a sequence of elementary row operations that transform $[A \mid I]$ to a reduced row echelon form. (Use $[i;c]$, $[i,j]$ and $[i,j;c]$ notation.) (Show work!)

Sol. $[A \mid I]$

$$\xrightarrow{[1,2]} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 3 & -2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{[2,1;2]} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & -3 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 3 & -2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{[4,1;-1]} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & -3 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{[3,4;1]} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 3 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{[4,2;-3]} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -7 & 0 & 1 \end{bmatrix} \xrightarrow{[1,3;2]} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 2 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -7 & 0 & 1 \end{bmatrix}$$

2. Write A as a product of elementary matrices $P(i;c)$, $P(i,j)$, $P(i,j;c)$.

Sol. Since $A^{-1} = P(1,3;2)P(4,2;-3)P(3,4;1)P(4,1;-1)P(2,1;2)P(1,2)$,

$$\begin{aligned} A &= P(1,2)^{-1}P(2,1;2)^{-1}P(4,1;-1)^{-1}P(3,4;1)^{-1}P(4,2;-3)^{-1}P(1,3;2)^{-1} \\ &= P(1,2)P(2,1;-2)P(4,1;1)P(3,4;-1)P(4,2;3)P(1,3;-2). \end{aligned}$$

3. Show that for a given $\boldsymbol{b}$, $A\boldsymbol{x} = \boldsymbol{b}$ always has a unique solution.

Sol. Since A is invertible, A^{-1} exists and if we set $\boldsymbol{x} = A^{-1}\boldsymbol{b}$, then

$$A\boldsymbol{x} = AA^{-1}\boldsymbol{b} = I\boldsymbol{b} = \boldsymbol{b}.$$

Hence $A^{-1}\boldsymbol{b}$ is a solution. Moreover, if $A\boldsymbol{x} = \boldsymbol{b}$, then $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$ and $\boldsymbol{x}$ is uniquely determined. Thus $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $A^{-1}\boldsymbol{b}$. ■

4. Find the solution $\boldsymbol{x}$ of an equation $A\boldsymbol{x} = \boldsymbol{c}$.

Sol. As above,

$$\boldsymbol{x} = A^{-1}\boldsymbol{c} = \begin{bmatrix} 0 & -1 & 2 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -3 & -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 4 \\ -3 \end{bmatrix}.$$