Solutions to Take-Home Quiz 6 (*October 22, 2010*)

Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and A be as follows.

$$\boldsymbol{u} = (1, 2, -1),\ \boldsymbol{v} = (3, 0, -2),\ \boldsymbol{w} = (5, -4, 6),\ \text{and } A = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix}.$$

1. Compute $\boldsymbol{u} \times \boldsymbol{v}$.

   Sol.
   $$\boldsymbol{u} \times \boldsymbol{v} = \left(\begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix}, - \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right) = (-4, -1, -6).$$

2. Find the volume of the parallelopiped (*heiko-6-mentai*) in 3-space determined by the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

   Sol. $|\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})| = |(5, -4, 6) \cdot (-4, -1, -6)| = |-20 + 4 - 36| = 52.$

   N.B. Do not forget to take the absolute value!

3. Find the characteristic polynomial of A and all eigenvalues of it.

   Sol.
   $$\begin{aligned} \det(xI - A) &= \begin{vmatrix} x & -1 & 0 \\ -6 & x-2 & -2 \\ 0 & -3 & x-4 \end{vmatrix} \\ &= x(x-2)(x-4) - 6x - 6(x-4) \\ &= x(x-2)(x-4) - 12(x-2) \\ &= (x-6)(x-2)(x+2), \qquad \text{or} \\ &= x^3 - 6x^2 - 4x + 24. \end{aligned}$$

   $6, 2$ and -2 are eigenvalues of A.

4. Find an eigenvector of A corresponding to its smallest eigenvalue.

   Sol. The smallest eigenvalue of A is -2. Find a nonzero vector $\boldsymbol{x}$ satisfying $((-2)I - A)\boldsymbol{x} = \boldsymbol{0}$ (or $A\boldsymbol{x} = (-2)\boldsymbol{x}$).

   $$\begin{bmatrix} -2 & -1 & 0 & 0 \\ -6 & -4 & -2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \overset{[2,1;-3]}{\longrightarrow} \begin{bmatrix} -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \overset{[2;-1]}{\longrightarrow} \begin{bmatrix} -2 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$
   $$\overset{[1,2;1]}{\longrightarrow} \begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \overset{[3,2;3]}{\longrightarrow} \begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{[1;-1/2]}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   The solutions are

   $$\boldsymbol{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \ (t \text{ is a parameter}), \qquad \text{and} \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

   is an eigenvector. (You may choose any non-zero t.)

   See Quiz 2. The matrix A is diagonalizable by a matrix T in it.