Solutions to Take-Home Quiz 7 (*October 29, 2010*)

$$\text{Let}\quad A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{e} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1. Find the characteristic polynomial $p(x)$ and all eigenvalues of A. (Solutions only!)

   Sol. $p(x) = x^3 - 8x^2 + 21x + 6$ and eigenvalues are $4, 3$ and 1.

$$\begin{aligned} \det(xI - A) &= \begin{vmatrix} x-3 & -1 & 0 \\ -1 & x-2 & -1 \\ 0 & -1 & x-3 \end{vmatrix} = (x-3)^2(x-2) - 2(x-3) \\ &= (x-3)(x^2 - 5x + 6 - 2) = (x-4)(x-3)(x-1). \end{aligned}$$

2. A is invertible. Why? (Use $p(x)$ only.)

   Sol. Since $p(0) \neq 0$, $0 \neq \det(-A) = (-1)^3 \det(A)$. Hence A is invertible.

3. Find an eigenvector corresponding to each of the eigenvalues of A. (Show work!)

   Sol. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be eigenvectors corresponding to $4, 3, 1$ respectively.

   Since each row sum is 4, if $\boldsymbol{v}_1 = [1, 1, 1]^T$, $A\boldsymbol{v}_1 = 4\boldsymbol{v}_1$. and $\boldsymbol{v}_1$ is an eigenvector corresponding to 4.

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

4. For a nonnegetive integer n, find $A^n\boldsymbol{e}$. (Show work!)

   Sol. Set $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 = \boldsymbol{e}$. Then

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & -3 & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix}.$$

   Thus $x_1 = 1/3$, $x_2 = -1/2$ and $x_3 = 1/6$ and $\boldsymbol{e} = \frac{1}{6}(2\boldsymbol{v}_1 - 3\boldsymbol{v}_2 + \boldsymbol{v}_3)$. Hence

$$A^n\boldsymbol{e} = \frac{1}{6}A^n(2\boldsymbol{v}_1 - 3\boldsymbol{v}_2 + \boldsymbol{v}_3) = \frac{1}{6}\left(2 \cdot 4^n \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3 \cdot 3^n \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 1^n \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right).$$

   Alt. Sol. Let $T = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$. Then $T^TT = \text{diag}(3, 2, 6)$. Hence $T^{-1} = \text{diag}(1/3, 1/2, 1/6)T^T$. Therefore if $D = \text{diag}(4, 3, 1)$, then $T^{-1}AT = D$. So

$$\begin{aligned} A^n\boldsymbol{e} &= TD^nT^{-1}\boldsymbol{e} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 1^n \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/2 & 0 & 1/2 \\ 1/6 & -2/6 & 1/6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \frac{1}{6} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \cdot 4^n \\ -3 \cdot 3^n \\ 1^n \end{bmatrix} = \frac{1}{6} \cdot \begin{bmatrix} 2 \cdot 4^n + 3 \cdot 3^n + 1 \\ 2 \cdot 4^n - 2 \\ 2 \cdot 4^n - 3 \cdot 3^n + 1 \end{bmatrix}. \end{aligned}$$