Take-Home **Quiz 1** (*Due at 7:00 p.m. on Fri. September 10, 2010*)

Division: **ID#:** **Name:**

Let us consider the following system of linear equations in 6 unknowns $x_1, x_2, \ldots, x_6$.

$$\begin{cases} -2x_1 + 2x_2 + 5x_3 + 4x_4 - 9x_5 - 10x_6 &= -13 \\ -x_1 + x_2 + 3x_3 + 3x_4 - 8x_5 - 7x_6 &= -6 \\ 2x_1 - 2x_2 - 4x_3 - x_4 - 3x_5 + 4x_6 &= 14 \\ x_1 - x_2 - 2x_3 - x_4 + x_5 + 3x_6 &= 7 \end{cases} \qquad B = \begin{bmatrix} 1 & -1 & -2 & -1 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 & -7 & -4 & 1 \\ 2 & -2 & -4 & -1 & -3 & 4 & 14 \\ -2 & 2 & 5 & 4 & -9 & -10 & -13 \end{bmatrix}$$

1. Find the augmented matrix A of the system of linear equations above.

2. The matrix B is obtained by applying elementary row operations twice to the augmented matrix A. Write the elementary row operation using the notation $[i;c]$, $[i,j]$, or $[i,j;c]$.

3. Find the reduced row echelon form of the augmented matrix A. (Solution only.)

4. Find the solution of the system of linear equations. Use parameters if necessary.

Message: (1) この授業に期待すること (2) あなたにとって数学とは [HP 掲載不可のときは明記のこと]

Solutions to Take-Home Quiz 1 (*September 10, 2010*)

$$\begin{cases} -2x_1 + 2x_2 + 5x_3 + 4x_4 - 9x_5 - 10x_6 &= -13 \\ -x_1 + x_2 + 3x_3 + 3x_4 - 8x_5 - 7x_6 &= -6 \\ 2x_1 - 2x_2 - 4x_3 - x_4 - 3x_5 + 4x_6 &= 14 \\ x_1 - x_2 - 2x_3 - x_4 + x_5 + 3x_6 &= 7 \end{cases} \qquad B = \begin{bmatrix} 1 & -1 & -2 & -1 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 & -7 & -4 & 1 \\ 2 & -2 & -4 & -1 & -3 & 4 & 14 \\ -2 & 2 & 5 & 4 & -9 & -10 & -13 \end{bmatrix}$$

1. Find the augmented matrix A of the system of linear equations above.

   Sol.

$$A = \begin{bmatrix} -2 & 2 & 5 & 4 & -9 & -10 & -13 \\ -1 & 1 & 3 & 3 & -8 & -7 & -6 \\ 2 & -2 & -4 & -1 & -3 & 4 & 14 \\ 1 & -1 & -2 & -1 & 1 & 3 & 7 \end{bmatrix}$$

2. The matrix B is obtained by applying elementary row operations twice to the augmented matrix A. Write the elementary row operation using the notation $[i; c]$, $[i, j]$, or $[i, j; c]$.

   Sol. First apply $[1, 4]$ and then $[2, 1; 1]$.

3. Find the reduced row echelon form of the augmented matrix A. (Solution only.)

   Sol. Apply the following consecutively in this order:

$$[3, 1; -2],\ [4, 1; 2],\ [4, 2; -1],\ [2, 3; -2],\ [1, 3; 1],\ [1, 2; 2].$$

   Then we have

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 2 & 1 & 9 \\ 0 & 0 & 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & -5 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   - There are many ways to obtain the reduced echelon form but the final matrix should be the same. When can we change the order of operations and when cannot?
   - Starting from the reduced row echelon form above, is it possible to obtain the matrix A back again by applying elementary row operations? Can you find the sequence of such elementary row operations from the one we obtained the reduced echelon form from A with a slight modification?

4. Find the solution of the system of linear equations. Use parameters if necessary.

   Sol.

$$\begin{cases} x_1 &= 9 + s - 2t - u \\ x_2 &= s \\ x_3 &= 1 - 3t \\ x_4 &= 5t + 2u \\ x_5 &= t \\ x_6 &= u \end{cases}, \text{ or } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

   s, t and u are parameters.

Take-Home **Quiz 2** (*Due at 7:00 p.m. on Fri. September 17, 2010*)

Division: **ID#:** **Name:**

Let L, T and C be matrices given below.

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 2 & -2 \\ 9 & -3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 6 & 2 & -2 & 0 & 1 & 0 \\ 9 & -3 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

1. Compute the product LT. (Show work!)

2. Find the reduced row echelon form of C.(Show work! Write operations as well in $[i;c], [i,j], [i,j;c]$ form.)

3. Compute $T^{-1}LT$. (Show work!)

Message 欄：(理系以外の人も含め) 高校・大学における数学は何のため？ [HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 2 (*September 17, 2010*)

Let L, T and C be matrices given below.

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 2 & -2 \\ 9 & -3 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 6 & 2 & -2 & 0 & 1 & 0 \\ 9 & -3 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

1. Compute the product LT. (Show work!)

   Sol.

$$LT = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 6 & 2 & -2 \\ 9 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 0+6+0 & 0+2+0 & 0-2+0 \\ 6+12+18 & 6+4-6 & 6-4+2 \\ 0+18+36 & 0+6-12 & 0-6+4 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -2 \\ 36 & 4 & 4 \\ 54 & -6 & -2 \end{bmatrix}$$

2. Find the reduced row echelon form of C. (Show work! Write operations as well in $[i;c], [i,j], [i,j;c]$ form.)

   Sol.

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 6 & 2 & -2 & 0 & 1 & 0 \\ 9 & -3 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{[2,1;-6]} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -8 & -6 & 1 & 0 \\ 9 & -3 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{[3,1;-9]} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -8 & -6 & 1 & 0 \\ 0 & -12 & -8 & -9 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{[3,2;-3]} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -8 & -6 & 1 & 0 \\ 0 & 0 & 16 & 9 & -3 & 1 \end{bmatrix} \xrightarrow{[2;-1/4]} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3/2 & -1/4 & 0 \\ 0 & 0 & 16 & 9 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow{[1,2;-1]} \begin{bmatrix} 1 & 0 & -1 & -1/2 & 1/4 & 0 \\ 0 & 1 & 2 & 3/2 & -1/4 & 0 \\ 0 & 0 & 16 & 9 & -3 & 1 \end{bmatrix} \xrightarrow{[3;-1/16]} \begin{bmatrix} 1 & 0 & -1 & -1/2 & 1/4 & 0 \\ 0 & 1 & 2 & 3/2 & -1/4 & 0 \\ 0 & 0 & 1 & 9/16 & -3/16 & 1/16 \end{bmatrix}$$

$$\xrightarrow{[1,4;-1]} \begin{bmatrix} 1 & 0 & 0 & 1/16 & 1/16 & 1/16 \\ 0 & 1 & 2 & 3/2 & -1/4 & 0 \\ 0 & 0 & 1 & 9/16 & -3/16 & 1/16 \end{bmatrix} \xrightarrow{[2,4;-2]} \begin{bmatrix} 1 & 0 & 0 & 1/16 & 1/16 & 1/16 \\ 0 & 1 & 0 & 3/8 & 1/8 & -1/8 \\ 0 & 0 & 1 & 9/16 & -3/16 & 1/16 \end{bmatrix}$$

3. Compute $T^{-1}LT$. (Show work!)

   Sol.

$$T^{-1}LT = \begin{bmatrix} 1/16 & 1/16 & 1/16 \\ 3/8 & 1/8 & -1/8 \\ 9/16 & -3/16 & 1/16 \end{bmatrix} \begin{bmatrix} 6 & 2 & -2 \\ 36 & 4 & 4 \\ 54 & -6 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

   Let D be the last matrix above, i.e., the diagonal matrix of size 3 with 6, 2, -2 on the diagonal. Since the columns of LT are scalar multiples of the columns of T by $6, 2, -2$ respectively, we have

$$LT = TD, \text{ or } T^{-1}LT = D.$$

   So this gives an alternate solution of this problem.

Take-Home **Quiz 3** (*Due at 7:00 p.m. on Fri. October 1, 2010*)

Division: **ID#:** **Name:**

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be as follows.

$$A = \begin{bmatrix} -2 & 1 & 4 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & -3 & 1 & -1 \\ 1 & 3 & -2 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{c} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

1. Find a sequence of elementary row operations that transform $[A \mid I]$ to a reduced row echelon form. (Use $[i;c]$, $[i,j]$ and $[i,j;c]$ notation.) (Show work!)

2. Write A as a product of elementary matrices $P(i;c)$, $P(i,j)$, $P(i,j;c)$.

3. Show that for a given $\boldsymbol{b}$, $A\boldsymbol{x} = \boldsymbol{b}$ always has a unique solution.

4. Find the solution $\boldsymbol{x}$ of an equation $A\boldsymbol{x} = \boldsymbol{c}$.

Message 欄：将来の夢、目標、25 年後の自分について、世界について。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 3 (*October 1, 2010*)

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be as follows.

$$A = \begin{bmatrix} -2 & 1 & 4 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & -3 & 1 & -1 \\ 1 & 3 & -2 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{c} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

1. Find a sequence of elementary row operations that transform $[A \mid I]$ to a reduced row echelon form. (Use $[i;c]$, $[i,j]$ and $[i,j;c]$ notation.) (Show work!)

   Sol. $[A \mid I]$

$$\overset{[1,2]}{\longrightarrow} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 3 & -2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{[2,1;2]}{\longrightarrow} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & -3 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 3 & -2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{[4,1;-1]}{\longrightarrow} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & -3 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \overset{[3,4;1]}{\longrightarrow} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 3 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\overset{[4,2;-3]}{\longrightarrow} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -7 & 0 & 1 \end{bmatrix} \overset{[1,3;2]}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 2 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -7 & 0 & 1 \end{bmatrix}$$

2. Write A as a product of elementary matrices $P(i;c)$, $P(i,j)$, $P(i,j;c)$.

   Sol. Since $A^{-1} = P(1,3;2)P(4,2;-3)P(3,4;1)P(4,1;-1)P(2,1;2)P(1,2)$,

$$\begin{aligned} A &= P(1,2)^{-1}P(2,1;2)^{-1}P(4,1;-1)^{-1}P(3,4;1)^{-1}P(4,2;-3)^{-1}P(1,3;2)^{-1} \\ &= P(1,2)P(2,1;-2)P(4,1;1)P(3,4;-1)P(4,2;3)P(1,3;-2). \end{aligned}$$

3. Show that for a given $\boldsymbol{b}$, $A\boldsymbol{x} = \boldsymbol{b}$ always has a unique solution.

   Sol. Since A is invertible, A^{-1} exists and if we set $\boldsymbol{x} = A^{-1}\boldsymbol{b}$, then

$$A\boldsymbol{x} = AA^{-1}\boldsymbol{b} = I\boldsymbol{b} = \boldsymbol{b}.$$

   Hence $A^{-1}\boldsymbol{b}$ is a solution. Moreover, if $A\boldsymbol{x} = \boldsymbol{b}$, then $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$ and $\boldsymbol{x}$ is uniquely determined. Thus $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $A^{-1}\boldsymbol{b}$. ■

4. Find the solution $\boldsymbol{x}$ of an equation $A\boldsymbol{x} = \boldsymbol{c}$.

   Sol. As above,

$$\boldsymbol{x} = A^{-1}\boldsymbol{c} = \begin{bmatrix} 0 & -1 & 2 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -3 & -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 4 \\ -3 \end{bmatrix}.$$

Take-Home **Quiz 4** (*Due at 7:00 p.m. on Sat. October 9, 2010*)

Division: **ID#:** **Name:**

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$ and B be the matrices given below.

$$A = \begin{bmatrix} -3 & 5 & 0 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -2 & 1 \\ -3 & 5 & 1 & 0 \\ 2 & 1 & 2 & -1 \\ -1 & 2 & 3 & 3 \end{bmatrix}.$$

1. Determine $\det(A)$ by cofactor expansion along the <u>third column</u>.

2. Find $\mathrm{adj}(A)$. (Solution only!)

3. Use Cramer's Rule to express x_3 as a quotient of two determinants for the equation $A\boldsymbol{x} = \boldsymbol{b}$, and evaluate x_3. (Solution only!)

4. Express $\det(B)$ by the cofactor expansion along the <u>first row</u> writing each of $C_{i,j}$ as a determinant. (Don't evaluate the determinant involved in $C_{i,j}$.)

5. Find $\det(B)$.

Message 欄： あなたにとって、豊かな生活とはどのようなものでしょうか。どのようなとき幸せだと感じますか。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 4 (*October 9, 2010*)

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$ and B be the matrices given below.

$$A = \begin{bmatrix} -3 & 5 & 0 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -2 & 1 \\ -3 & 5 & 1 & 0 \\ 2 & 1 & 2 & -1 \\ -1 & 2 & 3 & 3 \end{bmatrix}.$$

1. Determine $\det(A)$ by cofactor expansion along the <u>third column</u>.

   Sol. $\det(A) = A_{1,3}C_{1,3} + A_{2,3}C_{2,3} + A_{3,3}C_{3,3}$

$$= 0(-1)^{1+3}\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1)(-1)^{2+3}\begin{vmatrix} -3 & 5 \\ -1 & 2 \end{vmatrix} + 3(-1)^{3+3}\begin{vmatrix} -3 & 5 \\ 2 & 1 \end{vmatrix} = -1 - 39 = -40.$$

2. Find $\mathrm{adj}(A)$. (Solution only!)

   Sol. $C_{i,j} = (-1)^{i+j}M_{i,j}$ and $M_{i,j}$ is the determinant of the submatrix that remains after the ith row and jth column are deleted from A.

$$\mathrm{adj}(A) = \begin{bmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \\ C_{1,3} & C_{2,3} & C_{3,3} \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 5 & 0 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 0 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} -3 & 0 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & 5 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} -3 & 5 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & -15 & -5 \\ -5 & -9 & -3 \\ 5 & 1 & -13 \end{bmatrix}$$

3. Use Cramer's Rule to express x_3 as a quotient of two determinants for the equation $A\boldsymbol{x} = \boldsymbol{b}$, and evaluate x_3. (Solution only!)

   Sol.

$$x_3 = \frac{\begin{vmatrix} -3 & 5 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} -3 & 5 & 0 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{vmatrix}} = \frac{-32}{-40} = \frac{4}{5}$$

4. Express $\det(B)$ by the cofactor expansion along the <u>first row</u> writing each of $C_{i,j}$ as a determinant. (Don't evaluate the determinant involved in $C_{i,j}$.)

   Sol. $\det(B) = 1 \cdot C_{1,1} + 0 \cdot C_{1,2} + (-2) \cdot C_{1,3} + 1 \cdot C_{1,4}$

$$= \begin{vmatrix} 5 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 3 & 3 \end{vmatrix} - 2 \cdot \begin{vmatrix} -3 & 5 & 0 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{vmatrix} - \begin{vmatrix} -3 & 5 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 3 \end{vmatrix} = 40 - 2 \cdot (-40) - (-32) = 152.$$

5. Find $\det(B)$.

   Sol. See above.

Take-Home **Quiz 5** (*Due at 7:00 p.m. on Fri. October 15, 2010*)

Division: **ID#:** **Name:**

Let A and B be the 5×5 matrices given below.

$$A = \begin{bmatrix} 2 & 0 & -3 & 1 & 4 \\ -2 & 2 & -1 & -3 & 2 \\ -3 & -1 & 0 & 4 & 1 \\ 2 & 2 & 1 & 3 & -2 \\ -2 & -3 & 3 & -2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} a & b & b & b & b \\ b & a & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \\ b & b & b & b & a \end{bmatrix}.$$

1. Find the numbers x, y and z. (Solutions only.)

$$\det(A) = x \cdot \begin{vmatrix} 2 & 2 & 1 & 3 & -2 \\ 8 & 6 & 0 & 10 & -2 \\ 0 & 4 & 0 & 0 & 0 \\ -3 & -1 & 0 & 4 & 1 \\ -8 & -9 & 0 & -11 & 10 \end{vmatrix} = y \cdot \begin{vmatrix} 8 & 6 & 10 & -2 \\ 0 & 1 & 0 & 0 \\ -3 & -1 & 4 & 1 \\ -8 & -9 & -11 & 10 \end{vmatrix} = z \cdot \begin{vmatrix} 4 & 5 & -1 \\ -3 & 4 & 1 \\ -8 & -11 & 10 \end{vmatrix}.$$

$$x = \qquad , \quad y = \qquad , \quad z = \qquad .$$

2. Find $\det(A)$. (Show work.)

3. Explain why the following equalities hold.

$$\det(B) \overset{(1)}{=} \begin{vmatrix} a+4b & a+4b & a+4b & a+4b & a+4b \\ b & a & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \\ b & b & b & b & a \end{vmatrix} \overset{(2)}{=} (a+4b) \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ b & a & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \\ b & b & b & b & a \end{vmatrix}.$$

4. Find the condition that B is invertible.

Message 欄：数学（または他の科目）など何かを学んでいて感激したことについて。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 5 (*October 15, 2010*)

Let A and B be the 5×5 matrices given below.

$$A=\begin{bmatrix} 2 & 0 & -3 & 1 & 4\\ -2 & 2 & -1 & -3 & 2\\ -3 & -1 & 0 & 4 & 1\\ 2 & 2 & 1 & 3 & -2\\ -2 & -3 & 3 & -2 & 4\end{bmatrix},\quad B=\begin{bmatrix} a & b & b & b & b\\ b & a & b & b & b\\ b & b & a & b & b\\ b & b & b & a & b\\ b & b & b & b & a\end{bmatrix}.$$

1. Find the numbers x, y and z. (Solutions only.)

$$\det(A)=x\cdot\begin{vmatrix} 2 & 2 & 1 & 3 & -2\\ 8 & 6 & 0 & 10 & -2\\ 0 & 4 & 0 & 0 & 0\\ -3 & -1 & 0 & 4 & 1\\ -8 & -9 & 0 & -11 & 10\end{vmatrix}=y\cdot\begin{vmatrix} 8 & 6 & 10 & -2\\ 0 & 1 & 0 & 0\\ -3 & -1 & 4 & 1\\ -8 & -9 & -11 & 10\end{vmatrix}=z\cdot\begin{vmatrix} 4 & 5 & -1\\ -3 & 4 & 1\\ -8 & -11 & 10\end{vmatrix}.$$

$$x=-1,\qquad y=-4,\qquad z=-8.$$

2. Find $\det(A)$. (Show work.)

Sol.

$$\det(A)=-8\cdot\begin{vmatrix} 4 & 5 & -1\\ -3 & 4 & 1\\ -8 & -11 & 10\end{vmatrix}\overset{[1,2;1],[3,2;-10]}{=}-8\cdot\begin{vmatrix} 1 & 9 & 0\\ -3 & 4 & 1\\ 22 & -51 & 0\end{vmatrix}=8\begin{vmatrix} 1 & 9\\ 22 & -51\end{vmatrix}$$

$=8(-51-9\cdot 22)=-8\cdot 249=-1992.$ ∎

3. Explain why the following equalities hold.

$$\det(B)\overset{(1)}{=}\begin{vmatrix} a+4b & a+4b & a+4b & a+4b & a+4b\\ b & a & b & b & b\\ b & b & a & b & b\\ b & b & b & a & b\\ b & b & b & b & a\end{vmatrix}\overset{(2)}{=}(a+4b)\begin{vmatrix} 1 & 1 & 1 & 1 & 1\\ b & a & b & b & b\\ b & b & a & b & b\\ b & b & b & a & b\\ b & b & b & b & a\end{vmatrix}.$$

Sol. (1) Applied the following row operations. $[1,2;1]$, $[1,3;1]$, $[1,4;1]$, $[1,5;1]$ and the value of the determinant remains the same.

(2) Factor out $a+4b$ from the first row. ∎

4. Find the condition that B is invertible.

Sol. By applying the operations $[2,1;-b]$, $3,1;-b]$, $[4,1;-b]$, $[5,1;-b]$, we have

$$\det(B)=(a+4b)\begin{vmatrix} 1 & 1 & 1 & 1 & 1\\ b & a & b & b & b\\ b & b & a & b & b\\ b & b & b & a & b\\ b & b & b & b & a\end{vmatrix}=(a+4b)\begin{vmatrix} 1 & 1 & 1 & 1 & 1\\ 0 & a-b & 0 & 0 & 0\\ 0 & 0 & a-b & 0 & 0\\ 0 & 0 & 0 & a-b & 0\\ 0 & 0 & 0 & 0 & a-b\end{vmatrix}$$

$=(a+4b)(a-b)^4$. So B is invertible if and only if $(a+4b)(a-b)\neq 0$ if and only if $a+4b\neq 0$ and $a-b\neq 0$. ∎

Take-Home **Quiz 6** (*Due at 7:00 p.m. on Fri. October 22, 2010*)

Division: **ID#:** **Name:**

Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and A be as follows.

$$\boldsymbol{u} = (1, 2, -1),\ \boldsymbol{v} = (3, 0, -2),\ \boldsymbol{w} = (5, -4, 6),\ \text{and } A = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix}.$$

1. Compute $\boldsymbol{u} \times \boldsymbol{v}$. (Show work!)

2. Find the volume of the parallelopiped (*heiko-6-mentai*) in 3-space determined by the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. (Show work!)

3. Find the characteristic polynomial of A and all eigenvalues of it. (Show work!)

4. Find an eigenvector of A corresponding to its smallest eigenvalue. (Show work!)

Message 欄：あなたにとって一番たいせつな（または、たいせつにしたい）もの、ことはなんですか。そのほか、何でもどうぞ。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 6 (*October 22, 2010*)

Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and A be as follows.

$$\boldsymbol{u} = (1, 2, -1),\ \boldsymbol{v} = (3, 0, -2),\ \boldsymbol{w} = (5, -4, 6),\ \text{and } A = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix}.$$

1. Compute $\boldsymbol{u} \times \boldsymbol{v}$.

   Sol.
   $$\boldsymbol{u} \times \boldsymbol{v} = \left(\begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix}, - \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right) = (-4, -1, -6).$$

2. Find the volume of the parallelopiped (*heiko-6-mentai*) in 3-space determined by the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

   Sol. $|\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})| = |(5, -4, 6) \cdot (-4, -1, -6)| = |-20 + 4 - 36| = 52.$

   N.B. Do not forget to take the absolute value!

3. Find the characteristic polynomial of A and all eigenvalues of it.

   Sol.
   $$\begin{aligned} \det(xI - A) &= \begin{vmatrix} x & -1 & 0 \\ -6 & x-2 & -2 \\ 0 & -3 & x-4 \end{vmatrix} \\ &= x(x-2)(x-4) - 6x - 6(x-4) \\ &= x(x-2)(x-4) - 12(x-2) \\ &= (x-6)(x-2)(x+2), \quad \text{or} \\ &= x^3 - 6x^2 - 4x + 24. \end{aligned}$$

   $6, 2$ and -2 are eigenvalues of A.

4. Find an eigenvector of A corresponding to its smallest eigenvalue.

   Sol. The smallest eigenvalue of A is -2. Find a nonzero vector $\boldsymbol{x}$ satisfying $((-2)I - A)\boldsymbol{x} = \boldsymbol{0}$ (or $A\boldsymbol{x} = (-2)\boldsymbol{x}$).

   $$\begin{bmatrix} -2 & -1 & 0 & 0 \\ -6 & -4 & -2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \overset{[2,1;-3]}{\longrightarrow} \begin{bmatrix} -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \overset{[2;-1]}{\longrightarrow} \begin{bmatrix} -2 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

   $$\overset{[1,2;1]}{\longrightarrow} \begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \overset{[3,2;3]}{\longrightarrow} \begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{[1;-1/2]}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   The solutions are

   $$\boldsymbol{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \ (t \text{ is a parameter}), \quad \text{and} \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

   is an eigenvector. (You may choose any non-zero t.)

   See Quiz 2. The matrix A is diagonalizable by a matrix T in it.

Take-Home Quiz 7 (*Due at 7:00 p.m. on Fri. October 29, 2010*)

Division: **ID#:** **Name:**

$$\text{Let} \quad A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{e} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1. Find the characteristic polynomial $p(x)$ and all eigenvalues of A. (Solutions only!)

2. A is invertible. Why? (Use $p(x)$ only.)

3. Find an eigenvector corresponding to each of the eigenvalues of A. (Show work!)

4. For a nonnegetive integer n, find $A^n\boldsymbol{e}$. (Show work!)

Message 欄（何でもどうぞ）：ICU をどのようにして知りましたか。ICU をより魅力的にするにはどうしたらよいでしょうか。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 7 (*October 29, 2010*)

$$\text{Let} \quad A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{e} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1. Find the characteristic polynomial $p(x)$ and all eigenvalues of A. (Solutions only!)

   Sol. $p(x) = x^3 - 8x^2 + 21x + 6$ and eigenvalues are $4, 3$ and 1.

$$\begin{aligned} \det(xI - A) &= \begin{vmatrix} x-3 & -1 & 0 \\ -1 & x-2 & -1 \\ 0 & -1 & x-3 \end{vmatrix} = (x-3)^2(x-2) - 2(x-3) \\ &= (x-3)(x^2 - 5x + 6 - 2) = (x-4)(x-3)(x-1). \end{aligned}$$

2. A is invertible. Why? (Use $p(x)$ only.)

   Sol. Since $p(0) \neq 0$, $0 \neq \det(-A) = (-1)^3 \det(A)$. Hence A is invertible.

3. Find an eigenvector corresponding to each of the eigenvalues of A. (Show work!)

   Sol. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be eigenvectors corresponding to $4, 3, 1$ respectively.

   Since each row sum is 4, if $\boldsymbol{v}_1 = [1, 1, 1]^T$, $A\boldsymbol{v}_1 = 4\boldsymbol{v}_1$. and $\boldsymbol{v}_1$ is an eigenvector corresponding to 4.

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

4. For a nonnegetive integer n, find $A^n\boldsymbol{e}$. (Show work!)

   Sol. Set $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 = \boldsymbol{e}$. Then

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & -3 & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix}.$$

   Thus $x_1 = 1/3$, $x_2 = -1/2$ and $x_3 = 1/6$ and $\boldsymbol{e} = \frac{1}{6}(2\boldsymbol{v}_1 - 3\boldsymbol{v}_2 + \boldsymbol{v}_3)$. Hence

$$A^n\boldsymbol{e} = \frac{1}{6}A^n(2\boldsymbol{v}_1 - 3\boldsymbol{v}_2 + \boldsymbol{v}_3) = \frac{1}{6}\left(2 \cdot 4^n \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3 \cdot 3^n \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 1^n \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right).$$

   Alt. Sol. Let $T = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$. Then $T^TT = \mathrm{diag}(3, 2, 6)$. Hence $T^{-1} = \mathrm{diag}(1/3, 1/2, 1/6)T^T$. Therefore if $D = \mathrm{diag}(4, 3, 1)$, then $T^{-1}AT = D$. So

$$\begin{aligned} A^n\boldsymbol{e} &= TD^nT^{-1}\boldsymbol{e} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 1^n \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/2 & 0 & 1/2 \\ 1/6 & -2/6 & 1/6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \frac{1}{6} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \cdot 4^n \\ -3 \cdot 3^n \\ 1^n \end{bmatrix} = \frac{1}{6} \cdot \begin{bmatrix} 2 \cdot 4^n + 3 \cdot 3^n + 1 \\ 2 \cdot 4^n - 2 \\ 2 \cdot 4^n - 3 \cdot 3^n + 1 \end{bmatrix}. \end{aligned}$$