INTRODUCTION TO LINEAR ALGEBRA

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6 Determinants and Cofactor Expansion

Let us consider the following equations of 2×2 matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $ad - bc \neq 0$, the matrix on the left has its inverse. If $ad - bc = 0$, then it cannot have its inverse. Why?

Definition 6.1 Let $A = (a_{i,j})$ be a square matrix of size n. We define the *determinant* of A denoted by $\det(A)$ recursively as follows.

1. If $n = 1$ and $A = (a)$, then $\det(A) = a$.
2. Suppose $n > 1$ and the determinant of all square matrices of size $n-1$ are defined. Then for $1 \leq i, j \leq n$, the *minor of entry* $a_{i,j}$ is denoted by $M_{i.j}$ and is defined to be the determinant of the submatrix that remains after ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{i,j}$ is denoted by $C_{i,j}$ and is called the *cofactor of entry* $a_{i,j}$. Let
$$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + \cdots + a_{1,n}C_{1,n}.$$

Definition 6.2 Let $A = (a_{i,j})$ be a square matrix of size n. Then the left matrix is called the *matrix of cofactors* from A, and the matrix on the right that is the transpose of the left is called the *adjoint* of A and denoted by $\mathrm{adj}(A)$.

$$\tilde{A} = \begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n,1} & C_{n,2} & \cdots & C_{n,n} \end{bmatrix},$$

$$\mathrm{adj}(A) = \begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & C_{n,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,n} & C_{2,n} & \cdots & C_{n,n} \end{bmatrix}.$$

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Example 6.1 Let $n = 2$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ M_{1,1} = d, M_{1,2} = c, M_{2,1} = b, M_{2,2} = a.$$

$$\tilde{A} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}, \mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

and $\det(A) = ad - bc$.

Theorem 6.1 *Let $A = (a_{i,j})$ be a square matrix of size n and $\mathrm{adj}(A)$ the adjoint of A. Then*

$$A \cdot \mathrm{adj}(A) = \det(A)I = \mathrm{adj}(A) \cdot A.$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}\begin{bmatrix} C_{1,1} & \cdots & C_{j,1} & \cdots & C_{n,1} \\ C_{1,2} & \cdots & C_{j,2} & \cdots & C_{n,2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{1,n} & \cdots & C_{j,n} & \cdots & C_{n,n} \end{bmatrix}$$

$$= \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix}$$

Corollary 6.2 (2.1.1, 2.1.2) *Let $A = (a_{i,j})$ be a square matrix of size n and $C_{i,j}$ the cofactor entry of $a_{i,j}$. Then the following hold.*

(i) $\det(A) = a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n}$ *for* $i = 1, 2, \ldots, n$.

(ii) $\det(A) = a_{1,j}C_{1,j} + a_{2,j}C_{2,j} + \cdots + a_{n,j}C_{n,j}$ *for* $j = 1, 2, \ldots, n$.

(iii) $a_{i,1}C_{j,1} + a_{i,2}C_{j,2} + \cdots + a_{i,n}C_{j,n} = 0$ *for* $i, j = 1, 2, \ldots, n$ *with* $i \neq j$.

(iv) $a_{1,j}C_{1,i} + a_{2,j}C_{2,i} + \cdots + a_{n,j}C_{i,n} = 0$ *for* $i, j = 1, 2, \ldots, n$ *with* $i \neq j$.

(iv) *A is invertible if and only if* $\det(A) \neq 0$. *In this case*

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & C_{n,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,n} & C_{2,n} & \cdots & C_{n,n} \end{bmatrix}.$$

Remarks. At this point by computation it is possible to prove (i) (iii) of the corollary. But it is very difficult to prove the rest. As for (iv), it can be shown that $\det(A) \neq 0$ implies the invertibility of A but the converse is not yet possible.

7 Evaluation of Determinants

Review Let $A = (a_{i,j})$ be an $n \times n$ matrix.

1. $\text{adj}(A) = \tilde{A}^T = (C_{i,j})^T = ((-1)^{i+j}M_{i,j})^T$.
2. $A\text{adj}(A) = \det(A)I = \text{adj}(A)A$.
3. If $\det(A) \neq 0$, then A is invertible and $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$.
4. If $A\boldsymbol{x} = \boldsymbol{b}$ and $\det(A) \neq 0$, then $\boldsymbol{x} = A^{-1}\boldsymbol{b} = \frac{1}{\det(A)}\text{adj}(A)\boldsymbol{b}$.

Theorem 7.1 (Cramer's Rule (2.1.4)) *If $A\boldsymbol{x} = \boldsymbol{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. The solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)}, \ldots,\ x_n = \frac{\det(A_n)}{\det(A)},$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by $\boldsymbol{b}$.

Proof. Recall that if $C_{i,j}$ is a cofactor of entry $a_{i,j}$, and

$$\begin{aligned}\det(A) &= a_{1,j}C_{1,j} + a_{2,j}C_{2,j} + \cdots \\ &\quad + a_{n,j}C_{n,j} \text{ for } j = 1, 2, \ldots, n\end{aligned}$$

by Corollary 6.2 (iv).

$$\begin{aligned}\boldsymbol{x} &= A^{-1}\boldsymbol{b} \\ &= \frac{1}{\det(A)}\begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & C_{n,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,n} & C_{2,n} & \cdots & C_{n,n}\end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n\end{bmatrix} \\ &= \frac{1}{\det(A)}\begin{bmatrix} b_1C_{1,1} + b_2C_{2,1} + \cdots + b_nC_{n,1} \\ b_1C_{1,2} + b_2C_{2,2} + \cdots + b_nC_{n,2} \\ \cdots \\ b_1C_{1,n} + b_2C_{2,n} + \cdots + b_nC_{n,n}\end{bmatrix}.\end{aligned}$$

■

Definition 7.1 Let $A = (a_{i,j})$ be a square matrix of size n.

1. A is said to be an *upper triangular matrix* if $a_{i,j} = 0$ for all $i > j$.
2. A is said to be a *lower triangular matrix* if $a_{i,j} = 0$ for all $i < j$.
3. A is said to be a *diagonal matrix* if $a_{i,j} = 0$ for all $i \neq j$.

Theorem 7.2 (2.1.3, 2.2.2, 2.2.3) *Let $A = (a_{i,j})$ be a square matrix of size n.*

(i) $\det(A) = \det(A^T)$.

(ii) *If A is a triangular matrix (upper triangular lower triangular, or diagonal). then* $\det(A) = a_{1,1}a_{2,2}\cdots a_{n,n}$.

(iii) *The value of the determinant changes as follows by elementary row operations.*

(a) $A \xrightarrow{[i;c]} B \Rightarrow \det(B) = c\det(A)$, *and* $|P(i;c)A| = |P(i;c)||A| = c|A|$.

(b) $A \xrightarrow{[i,j]} B \Rightarrow \det(B) = -\det(A)$, *and* $|P(i,j)A| = |P(i,j)||A| = -|A|$.

(c) $A \xrightarrow{[i,j;c]} B \Rightarrow \det(B) = \det(A)$, *and* $|P(i,j;c)A| = |P(i,j;c)||A| = |A|$.

Similar results hold for elementary column operations by Theorem 7.2 (i).

8 Properties of the Determinant Function

Let $A = (a_{i,j})$ be an $n \times n$ square matrix. We write

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ & \cdots\cdots\cdots & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n}\end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m\end{bmatrix},$$

where $\boldsymbol{a}_i = [a_{i,1}, a_{i,2}, \ldots, a_{i,n}]$.

Proposition 8.1 *Let $A = (a_{i,j})$ be an $n \times n$ square matrix and $\boldsymbol{a}_i$ its i-th row. Then the following hold.*

(i) *For a constant c,*

$$c\det(A) = \det\begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ c \cdot \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_n\end{bmatrix},$$

$$\det\begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i + \boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_n\end{bmatrix} = \det\begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_n\end{bmatrix} + \det\begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i' \\ \vdots \\ \boldsymbol{a}_n\end{bmatrix}.$$

(ii) *If $\boldsymbol{a}_i = \boldsymbol{a}_j$ for some $i \neq j$, then* $\det(A) = 0$.

Proof. (i) Straightforward by cofactor expansion along the i-th row.

(ii) Use induction. ■

Remarks. Let D be a function defined for each square matrix of size n. If $D(I) = 1$, the property (ii) above is satisfied and

$$c \cdot D(A) = D\left(\begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ c \cdot \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}\right),$$

$$D\left(\begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i + \boldsymbol{a}'_i \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}\right) = D\left(\begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}\right) + D\left(\begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}'_i \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}\right).$$

Then $D(A) = \det(A)$ for all matrices A.

Theorem 8.2 (2.2.3) *Let $A = (a_{i,j})$ be a square matrix of size n. The value of the determinant changes as follows by elementary row operations.*

(a) $A \xrightarrow{[i;c]} B \Rightarrow \det(B) = c\det(A)$, *and* $|P(i;c)A| = |P(i;c)||A| = c|A|$.

(b) $A \xrightarrow{[i,j]} B \Rightarrow \det(B) = -\det(A)$, *and* $|P(i,j)A| = |P(i,j)||A| = -|A|$.

(c) $A \xrightarrow{[i,j;c]} B \Rightarrow \det(B) = \det(A)$, *and* $|P(i,j;c)A| = |P(i,j;c)||A| = |A|$.

In particular, if P is an elementary matrix, $|PA| = |P||A|$.

Proof. (a) Straightforward from Proposition 8.1 (i).

(b) We have from the following.

$$0 = \begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i + \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_i + \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{vmatrix} = \begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{vmatrix} + \begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_n \end{vmatrix}.$$

(c) We have from the following.

$$\begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i + c \cdot \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{vmatrix} = \begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{vmatrix} + c \cdot \begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \cdot \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{vmatrix} = \begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{vmatrix}.$$

■

Theorem 8.3 (2.3.3) *A square matrix is invertible if and only if* $\det(A) \neq 0$.

Proof. If $\det(A) \neq 0$, then A is invertible. Conversely if A is invertible, A can be written as a product of elementary matrices. ■

Theorem 8.4 (2.3.1) *Let A and B be square matrices of size n. Then*

$$\det(AB) = \det(A)\det(B).$$

Proof. If A is not invertible, then by Theorem **??**, AB is not invertible. Hence $\det(AB) = 0 = \det(A)\det(B)$ by Theorem 8.3. On the other hand, if A is invertible by Theorem **??**, A is a product of elementary matrices. Let $A = P_1 P_2 \cdots P_\ell$. Now by consecutive applications of Theorem 8.2,

$$\begin{aligned} |AB| &= |P_1 P_2 \cdots P_\ell B| = |P_1||P_2 \cdots P_\ell B| \\ &= |P_1||P_2| \cdots |P_\ell||B| = |P_1 P_2 \cdots P_\ell||B| \\ &= |A||B|. \end{aligned}$$

This proves the assertion. ■

Example 8.1 The following is called the Vandermonde's determinant.

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ & & \cdots\cdots\cdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{i>j}(x_i - x_j) = \prod_{j=1}^{n-1}\prod_{i=j+1}^{n}(x_i - x_j).$$

9 A Combinatorial Approach to Determinant

$$\begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i + \boldsymbol{a}'_i \\ \vdots \\ \boldsymbol{a}_m \end{vmatrix} = \begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_m \end{vmatrix} + \begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}'_i \\ \vdots \\ \boldsymbol{a}_m \end{vmatrix}, \quad \begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ c\boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_m \end{vmatrix} = c\begin{vmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_m \end{vmatrix},$$

$$\text{where} \quad \begin{aligned} \boldsymbol{a}_\ell &= [a_{\ell,1}, a_{\ell,2}, \ldots, a_{\ell,n}] \\ & j = 1, 2, \ldots, n, \\ \boldsymbol{a}'_i &= [a'_{i,1}, a'_{i,2}, \ldots, a'_{i,n}] \\ & c \text{ a constant.} \end{aligned}$$

Size Two

$$\begin{aligned}
&\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \\
&= \begin{vmatrix} a_{1,1} & 0 \\ a_{2,1} & a_{2,2} \end{vmatrix} + \begin{vmatrix} 0 & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \\
&= \begin{vmatrix} a_{1,1} & 0 \\ a_{2,1} & 0 \end{vmatrix} + \begin{vmatrix} a_{1,1} & 0 \\ 0 & a_{2,2} \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & a_{1,2} \\ a_{2,1} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{1,2} \\ 0 & a_{2,2} \end{vmatrix} \\
&= a_{1,1}a_{2,2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + a_{1,2}a_{2,1} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\
&= (+1)a_{1,1}a_{2,2} + (-1)a_{1,2}a_{2,1}.
\end{aligned}$$

Size Three: Formula of Sarras

$$\begin{aligned}
&\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \\
&= a_{1,1}a_{2,2}a_{3,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&\quad + a_{1,1}a_{2,3}a_{3,2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\
&\quad + a_{1,2}a_{2,1}a_{3,3} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&\quad + a_{1,2}a_{2,3}a_{3,1} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\
&\quad + a_{1,3}a_{2,1}a_{3,2} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\
&\quad + a_{1,3}a_{2,2}a_{3,1} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \\
&= (+1)a_{1,1}a_{2,2}a_{3,3} + (-1)a_{1,1}a_{2,3}a_{3,2} \\
&\quad + (-1)a_{1,2}a_{2,1}a_{3,3} + (+1)a_{1,2}a_{2,3}a_{3,1} \\
&\quad + (+1)a_{1,3}a_{2,1}a_{3,2} + (-1)a_{1,3}a_{2,2}a_{3,1} \\
&= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\
&\quad - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}.
\end{aligned}$$

Example 9.1 By Sarras,

$$\begin{aligned}
&\begin{vmatrix} 0 & 1 & 0 \\ 6 & 1 & 3 \\ 0 & 4 & 3 \end{vmatrix} \\
&= 0 \cdot 1 \cdot 3 + 1 \cdot 3 \cdot 8 + 0 \cdot 6 \cdot 4 \\
&\quad - 0 \cdot 1 \cdot 0 - 1 \cdot 6 \cdot (-2) - 0 \cdot 3 \cdot 4 \\
&= -18.
\end{aligned}$$

Definition 9.1 A *permutation* of the set of integers $\{1, 2, \ldots, n\}$ is an arrangement of these integers in some order without omissions or repetitions. Let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. Let $\sigma = (i_1, i_2, \ldots, i_n)$ be a permutation. Then the number of inversions, denoted by $\ell(\sigma)$, is defined by

$$\ell(\sigma) = |\{(j,k) \mid j < k,\ i_j > i_k\}|.$$

The *signature* of σ, denoted by $\text{sign}(\sigma)$, is defined by

$$\text{sign}(\sigma) = (-1)^{\ell(\sigma)}.$$

A permutation is called *even* if the total number of inversions, i.e., $\ell(\sigma)$ is an even integer, and is called *odd* if the total number of inversions is an odd integer.

Theorem 9.1 *Let $A = (a_{i,j})$ be a square matrix of size n. Then*

$$\begin{aligned}
&\det(A) \\
&= \sum_{(i_1,i_2,\ldots,i_n) \in S_n} \text{sign}((i_1, i_2, \ldots, i_n)) a_{1,i_1} a_{2,i_2} \cdots a_{n,i_n} \\
&= \sum_{(i_1,i_2,\ldots,i_n) \in S_n} (-1)^{\ell((i_1,i_2,\ldots,i_n))} a_{1,i_1} a_{2,i_2} \cdots a_{n,i_n}.
\end{aligned}$$

Proof. Let $(i_1, i_2, \ldots, i_n) \in S_n$ and $P(i_1, i_2, \ldots, i_n)$ be a square matrix of size n such that (j, i_j) entry is 1 and 0 otherwise. Then

$$\begin{aligned}
&\det(A) \\
&= \sum_{(i_1,i_2,\ldots,i_n) \in S_n} a_{1,i_1} a_{2,i_2} \cdots a_{n,i_n} |P(i_1, i_2, \ldots, i_n)| \\
&= \sum_{(i_1,i_2,\ldots,i_n) \in S_n} (-1)^{\ell((i_1,i_2,\ldots,i_n))} a_{1,i_1} a_{2,i_2} \cdots a_{n,i_n}.
\end{aligned}$$

■

10 Equivalent Conditions

Theorem 10.1 (2.3.6) *If A is an $n \times n$ matrix, then the following are equivalent.*

(a) *A is invertible.*

(b) *$A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution, i.e., $\boldsymbol{x} = \boldsymbol{0}$.*

(c) *The reduced row-echelon form of A is I_n.*

(d) *A is expressible as a product of elementary matrices.*

(e) *$A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$.*

(f) *$A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $n \times 1$ matrix $\boldsymbol{b}$.*

(g) $\det(A) \neq 0$.

Exercise 10.1 What are the negations of the conditions above?