INTRODUCTION TO LINEAR ALGEBRA

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10 Vectors in 2-Space and 3-Space

Definition 10.1 1. Vectors: $\boldsymbol{v} = \overrightarrow{PQ}$. $P = P(p_1, p_2, p_3)$: initial point, $Q = Q(q_1, q_2, q_3)$: terminal point.

2. Components of $\boldsymbol{v} = (v_1, v_2, v_3)$ (or $\boldsymbol{v} = (v_1, v_2)$). Initial point at the origin.

   $\boldsymbol{v} = \overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2, q_3 - p_3)$.

3. $\boldsymbol{v} = (v_1, v_2, v_3)$, $\boldsymbol{w} = (w_1, w_2, w_3)$. $k \in \boldsymbol{R}$. Then $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$ and $k\boldsymbol{v} = (kv_1, kv_2, kv_3)$.

4. $\|\boldsymbol{v}\| = \sqrt{v_2^2 + v_2^2 + v_3^2}$: the norm of $\boldsymbol{v}$.

5. Let $P = P(p_1, p_2, p_3)$, $Q = Q(q_1, q_2, q_3)$. Then $d(P, Q) = \|\overrightarrow{PQ}\|$ is the distance.

6. Let $\boldsymbol{v} = (v_1, v_2, v_3)$, $\boldsymbol{w} = (w_1, w_2, w_3)$. Then $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 (= \boldsymbol{v}\boldsymbol{w}^T)$: Euclidean inner product. In particular, $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$.

   $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$, $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$, $(k\boldsymbol{u}) \cdot \boldsymbol{v} = k(\boldsymbol{u} \cdot \boldsymbol{v})$.

Example 10.1 $\boldsymbol{u} = (3, -2, -5)$, $\boldsymbol{v} = (1, 4, -4)$, $\boldsymbol{w} = (0, 3, 2)$.

Note.

1. The vectors above are often called *row vectors*. *Column vectors* are also considered.

2. We can extend the definitions above to vectors in $\boldsymbol{R}^n$ and Cacchy-Schwarz is valid for vectors in $\boldsymbol{R}^n = \{(v_1, v_2, \ldots, v_n) \mid v_i \in \boldsymbol{R}\}$.

Theorem 10.1 (Cauchy-Schwarz) *The following holds for* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^n$.

$$-\|\boldsymbol{u}\|\|\boldsymbol{v}\| \leq \boldsymbol{u} \cdot \boldsymbol{v} \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

Proof. Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ be non-zero vectors in $\boldsymbol{R}^n$.

1. Let λ be a real number. Show the following. (Hint: use $\|\boldsymbol{w}\|^2 = \boldsymbol{w} \cdot \boldsymbol{w}$.)

$$\|\lambda\boldsymbol{u} + \boldsymbol{v}\|^2 = \lambda^2\|\boldsymbol{u}\|^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v})\lambda + \|\boldsymbol{v}\|^2.$$

2. Using the fact that $\|\lambda\boldsymbol{u} + \boldsymbol{v}\|^2 \geq 0$ for all real λ and a property of a quadratic function, show the Cauchy-Schwarz Inequality. (Hint: Discriminant (*Hanbetsu-shiki*))

3. Show the equivalence of the following:

$$|\boldsymbol{u} \cdot \boldsymbol{v}| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \quad \Leftrightarrow \quad \text{There exists } \alpha \in \boldsymbol{R} \text{ such that } \boldsymbol{u} = \alpha\boldsymbol{v}.$$

Definition 10.2 1. Suppose $\boldsymbol{u} \neq \boldsymbol{0}$ and $\boldsymbol{v} \neq \boldsymbol{0}$. The angle θ such that $0 \leq \theta \leq \pi$ satisfying

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}.$$

2. Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal whenever $\boldsymbol{u} \cdot \boldsymbol{v} = 0$, i.e., $\theta = \pi/2$.

3. Let $\boldsymbol{u}, \boldsymbol{v} \in V$ and $\boldsymbol{v} \neq \boldsymbol{0}$. Then there exist $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ such that

$$\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2, \ \boldsymbol{u}_1 = \alpha\boldsymbol{v}, \text{ and } \boldsymbol{u}_2 \cdot \boldsymbol{v} = 0.$$

   (a) $\text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \frac{\boldsymbol{v} \cdot \boldsymbol{u}}{\|\boldsymbol{v}\|^2}\boldsymbol{v}$: vector component of $\boldsymbol{u}$ along $\boldsymbol{v}$.

   (b) $\boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \boldsymbol{u} - \frac{\boldsymbol{v} \cdot \boldsymbol{u}}{\|\boldsymbol{v}\|^2}\boldsymbol{v}$: vector component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{v}$.

4. Let $\boldsymbol{u}, \boldsymbol{v}$ be vectors in $\boldsymbol{R}^3$. Then the *cross product* of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as follows.

$$\begin{aligned} \boldsymbol{u} \times \boldsymbol{v} & \\ &= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \\ &= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= -\boldsymbol{v} \times \boldsymbol{u}, \end{aligned}$$

   where $\boldsymbol{i} = (1, 0, 0)$, $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$.

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5. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\boldsymbol{R}^3$. Then the *scalar triple product* of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ is defined as follows.

$$\begin{aligned}
&\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) \\
&= (u_1, u_2, u_3) \cdot \\
&\quad \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
&= \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})
\end{aligned}$$

Theorem 10.2 *Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\boldsymbol{R}^3$*

1. $\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2$. *(Lagrange's identity)*

2. $\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \sin\theta$. *The area of thee parallelogram determined by $\boldsymbol{u}$ and $\boldsymbol{v}$.*

3. $\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0$.

4. *Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, and $\boldsymbol{u}$ is rotated through the angle θ until it coincides with $\boldsymbol{v}$. If the fingers of the right hand are cupped so that they point in the direction of rotation., then the thumb indicates the direction of $\boldsymbol{u} \times \boldsymbol{v}$.*

5. $|\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|$ *is the volume of the parallelopiped determined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.*

Proof.

1. By computation.
2. $\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 (1 - \cos^2\theta) = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 \sin^2\theta$
3. Clear by definition.
4. Check special cases.
5. Clear by above.

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Theorem 10.3 *Let $P_0 = P_0(x_0, y_0, z_0)$ be a point and $\boldsymbol{n} = (a, b, c)$ a vector.*

1. $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$*: point normal form of the equation of a plane $P = P(x, y, z)$.*

2. *Planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ are parallel if and only if (a, b, c) is a nonzero scalar times (a', b', c').*

3. *The distance D from a point $P(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is*

$$D = \frac{|ax_0 + by_0 + cz_0|}{\sqrt{a^2 + b^2 + c^2}}.$$