Linear Algebra II *February 23, 2007*

Final Exam 2006/7

(Total: 100pts)

Division: **ID#:** **Name:**

In the following if you use a theorem, state it. As for the following theorem, state which item (a) - (d) is applied.

Theorem 1 *Let V be an n-dimensional vector space, and S a set of vectors in V.*

(a) *Suppose S has exactly n vectors. Then S is linearly independent if and only if S spans V.*

(b) *If S spans V but not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(c) *If S is linearly independent that is not already a basis for V, then S can be enlarged to a basis of V by inserting appropriate vectors into S.*

(d) *If W is a subspace of V, then $\dim(W) \leq \dim(V)$. Moreover if $\dim(W) = \dim(V)$, then $W = V$.*

1. Let V be an inner product space. Suppose $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a set of nonzero orthogonal vectors in V.

   (a) Show that S is a linearly independent set. (10 pts)

Points:

1. (*a*)	(*b*)	(*c*)	(*d*)	2. (*a*)	(*b*)	(*c*)	3. (*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)

Message: (1) 数学について (2) この授業について特に改善点について (3) その他何でもどうぞ（裏面も使って下さい）[HP 掲載不可のときは明記のこと]

(b) In addition assume that $\|\boldsymbol{v}_1\| = \|\boldsymbol{v}_2\| = \|\boldsymbol{v}_3\| = \|\boldsymbol{v}_4\| = \sqrt{2}$. Evaluate (5 pts)

$$\left\| \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 + \boldsymbol{v}_4) \right\|.$$

(c) Find the rank of the following matrix (5 pts)

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

(d) Determine whether the following set of vectors is linearly independent. (5 pts)

$$\{\boldsymbol{v}_1 - \boldsymbol{v}_2,\ \boldsymbol{v}_2 - \boldsymbol{v}_3,\ \boldsymbol{v}_3 - \boldsymbol{v}_4,\ \boldsymbol{v}_4 - \boldsymbol{v}_1\}.$$

2. Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_s\}$ be a linearly independent set of vectors in V and $\boldsymbol{w}_1 = T(\boldsymbol{v}_1), \boldsymbol{w}_2 = T(\boldsymbol{v}_2), \dots, \boldsymbol{w}_s = T(\boldsymbol{v}_s)$. Let

$$U = \mathrm{Ker}(T) = \{\boldsymbol{v} \in V \mid T(\boldsymbol{v}) = \boldsymbol{0}\}.$$

(a) Show that U is a subspace of V. (5 pts)

(b) Suppose that $\boldsymbol{v} \in V$ satisties that $\boldsymbol{v} \notin \mathrm{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_s)$. Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_s, \boldsymbol{v}\}$ is linearly independent. (10 pts)

(c) Suppose that $\mathrm{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_s) \cap U = \{\boldsymbol{0}\}$. Show that $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_s\}$ is linearly independent. (10 pts)

3. Let $T : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be a linear transformation, $A = [T]$ the standard matrix of T given below, and $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_2$ the column vectors of A.

$$A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3] = \begin{bmatrix} 0 & 6 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}, \ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 9 \\ -3 \\ 1 \end{bmatrix}.$$

We consider four sets of vectors: $B = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, the standard basis of $\boldsymbol{R}^3$, $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are given above, $S = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$, the set of column vectors of A, and $S' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, the orthonormal basis of $\boldsymbol{R}^3$ to be constructed in (b). You may use the fact that B is actually a basis of $\boldsymbol{R}^3$ and $A\boldsymbol{v}_1 = 6\boldsymbol{v}_1$, $A\boldsymbol{v}_2 = 2\boldsymbol{v}_2$ and $A\boldsymbol{v}_3 = -2\boldsymbol{v}_3$. To give your answer, show work and give your reason.

(a) Show that $S = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ is a basis of $\boldsymbol{R}^3$. (10 pts)

(b) Using the basis S, find an orthonormal basis $S' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ of $\boldsymbol{R}^3$ with respect to the usual Euclidean inner product by the Gram-Schmidt process. Note that $\boldsymbol{u}_1 = \boldsymbol{a}_1 = \boldsymbol{e}_2$. (10 pts)

(c) Express each of $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ as a linear combination of the orthonormal basis S'. (5 pts)

(d) Find $[I]_{S',B}$ and $[I]_{B,S'}$, where $I : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ ($\boldsymbol{x} \mapsto \boldsymbol{x}$) is the identity operator on $\boldsymbol{R}^3$. (10 pts)

(e) Express $[T]_{S'}$ using A, $[I]_{S',B}$ and $[I]_{B,S'}$. (5 pts)

(f) Find the matrix $[T]_{B'}$ for T with respect to the basis $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. (10 pts)

Linear Algebra II *February 23, 2007*

Solutions to Final Exam 2006/7

In the following if you use a theorem, state it. As for the following theorem, state which item (a) - (d) is applied.

Theorem 1 *Let V be an n-dimensional vector space, and S a set of vectors in V.*

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(d) *If W is a subspace of V, then $\dim(W) \le \dim(V)$. Moreover if $\dim(W) = \dim(V)$, then $W = V$.*

1. Let V be an inner product space. Suppose $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a set of nonzero orthogonal vectors in V.

   (a) Show that S is a linearly independent set. (10 pts)
   Sol. Suppose

   $$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_n\boldsymbol{v}_n = \mathbf{0}.$$

   For each $i = 1, 2, \ldots, n$,

   $$\begin{aligned} 0 &= \langle \mathbf{0}, \boldsymbol{v}_i \rangle \\ &= \langle k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_n\boldsymbol{v}_n, \boldsymbol{v}_i \rangle \qquad \text{(by the equation above)} \\ &= k_1\langle \boldsymbol{v}_1, \boldsymbol{v}_i \rangle + k_2\langle \boldsymbol{v}_2, \boldsymbol{v}_i \rangle + \cdots + k_n\langle \boldsymbol{v}_n, \boldsymbol{v}_i \rangle \qquad \text{(by linearlity)} \\ &= k_i\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle. \qquad (S \text{ is a set of orthogonal vectors}) \end{aligned}$$

   Since $\boldsymbol{v}_i \neq \mathbf{0}$ by definition, $\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle > 0$ (by the definition of inner product). Hence $k_i\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle = 0$ implies that $k_i = 0$.
   Therefore $k_1 = k_2 = \cdots = k_n = 0$ and S is linearly independent. ■

   (b) In addition assume that $\|\boldsymbol{v}_1\| = \|\boldsymbol{v}_2\| = \|\boldsymbol{v}_3\| = \|\boldsymbol{v}_4\| = \sqrt{2}$. Evaluate (5 pts)

   $$\left\| \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 + \boldsymbol{v}_4) \right\|.$$

   Sol.

   $$\begin{aligned} \left\| \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 + \boldsymbol{v}_4) \right\|^2 &= \langle \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 + \boldsymbol{v}_4), \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 + \boldsymbol{v}_4) \rangle \\ &= \frac{1}{4}(\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle + \langle \boldsymbol{v}_2, \boldsymbol{v}_2 \rangle + \langle \boldsymbol{v}_3, \boldsymbol{v}_3 \rangle + \langle \boldsymbol{v}_4, \boldsymbol{v}_4 \rangle)(S \text{ is an orthogonal set}) \\ &= \frac{1}{4}(2 + 2 + 2 + 2) = 2. \qquad (\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle = \|\boldsymbol{v}_i\|^2 = 2 \text{ for } i = 1, 2, 3, 4) \end{aligned}$$

   Hence $\left\| \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 + \boldsymbol{v}_4) \right\| = \sqrt{2}$. ■

(c) Find the rank of the following matrix (5 pts)

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Sol. Let $\boldsymbol{j} = [1, 1, 1, 1]^T$. Then $A\boldsymbol{j} = \mathbf{0}$. Hence $\boldsymbol{j}$ is in the nullspace and nullity$(A) \geq 1$, and rank$(A) = 4 - \text{nullity}(A) \leq 3$. Since

$$\det \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = 1 \neq 0, \quad \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and hence

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a linearly independent set. Therefore rank$(A) \geq 3$ and rank$(A) = 3$. ■

(d) Determine whether the following set of vectors is linearly independent. (5 pts)

$$\{\boldsymbol{v}_1 - \boldsymbol{v}_2,\ \boldsymbol{v}_2 - \boldsymbol{v}_3,\ \boldsymbol{v}_3 - \boldsymbol{v}_4,\ \boldsymbol{v}_4 - \boldsymbol{v}_1\}.$$

Sol. Since

$$(\boldsymbol{v}_1 - \boldsymbol{v}_2) + (\boldsymbol{v}_2 - \boldsymbol{v}_3) + (\boldsymbol{v}_3 - \boldsymbol{v}_4) + (\boldsymbol{v}_4 - \boldsymbol{v}_1) = \mathbf{0},$$

it is linearly dependent. ■

Note that if $W = \text{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4)$, $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ and $T : W \to W$ is a linear transformation such that $T(\boldsymbol{v}_1) = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $T(\boldsymbol{v}_2) = \boldsymbol{v}_2 - \boldsymbol{v}_3$, $T(\boldsymbol{v}_3) = \boldsymbol{v}_3 - \boldsymbol{v}_4$ and $T(\boldsymbol{v}_4) = \boldsymbol{v}_4 - \boldsymbol{v}_1$, then $A = [T]_B$, i.e., A is the matrix for T with respect to the basis B. Hence rank$(T) = \text{rank}(A) = 3 < 4$ and the set is linearly dependent. Check the detail.

2. Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s\}$ be a linearly independent set of vectors in V and $\boldsymbol{w}_1 = T(\boldsymbol{v}_1), \boldsymbol{w}_2 = T(\boldsymbol{v}_2), \ldots, \boldsymbol{w}_s = T(\boldsymbol{v}_s)$. Let

$$U = \text{Ker}(T) = \{\boldsymbol{v} \in V \mid T(\boldsymbol{v}) = \mathbf{0}\}.$$

(a) Show that U is a subspace of V. (5 pts)

Sol. Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in U$. Then by the definition of U, $T(\boldsymbol{u}_1) = T(\boldsymbol{u}_2) = \mathbf{0}$. Since T is a linear transformation,

$$T(\boldsymbol{u}_1 + \boldsymbol{u}_2) = T(\boldsymbol{u}_1) + T(\boldsymbol{u}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0},\ T(k\boldsymbol{u}_1) = kT(\boldsymbol{u}_1) = k\mathbf{0} = \mathbf{0}.$$

Hence $\boldsymbol{u}_1 + \boldsymbol{u}_2 \in U$ and $k\boldsymbol{u}_1 \in U$ as these vectors satisfy the condition defining U. Therefore U is a subspace (by Theorem 3.2 (5.2.1)). ■

(b) Suppose that $\boldsymbol{v} \in V$ satisties that $\boldsymbol{v} \notin \text{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s)$. Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s, \boldsymbol{v}\}$ is linearly independent. (10 pts)

Sol. Suppose

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_s\boldsymbol{v}_s + k\boldsymbol{v} = \mathbf{0}.$$

We need to show that $k = k_1 = k_2 = \cdots = k_s = 0$. If $k \neq 0$, then

$$\boldsymbol{v} = (-k_1/k)\boldsymbol{v}_1 + (-k_2/k)\boldsymbol{v}_2 + \cdots + (-k_s/k)\boldsymbol{v}_s \in \text{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s).$$

This is against our hypothesis. Hence $k = 0$, and

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_s\boldsymbol{v}_s = \mathbf{0}.$$

Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s\}$ is linearly independent, $k_1 = k_2 = \cdots = k_s = 0$. Therefore $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s, \boldsymbol{v}\}$ is linearly independent. ■

(c) Suppose that $\text{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s) \cap U = \{\mathbf{0}\}$. Show that $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s\}$ is linearly independent. (10 pts)

Sol. Suppose

$$k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2 + \cdots + k_s\boldsymbol{w}_s = \mathbf{0}.$$

Since $\boldsymbol{w}_1 = T(\boldsymbol{v}_1), \boldsymbol{w}_2 = T(\boldsymbol{v}_2), \ldots, \boldsymbol{w}_s = T(\boldsymbol{v}_s)$,

$$\begin{aligned} \mathbf{0} &= k_1T(\boldsymbol{v}_1) + k_2T(\boldsymbol{v}_2) + \cdots + k_sT(\boldsymbol{v}_s) \\ &= T(k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_s\boldsymbol{v}_s). \end{aligned}$$

Hence

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_s\boldsymbol{v}_s \in \text{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s) \cap U = \{\mathbf{0}\}.$$

This implies that

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_s\boldsymbol{v}_s = \mathbf{0}.$$

Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_s\}$ is a linearly independent set, $k_1 = k_2 = \cdots = k_s = 0$. Therefore $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s\}$ is linearly independent. ■

3. Let $T : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be a linear transformation, $A = [T]$ the standard matrix of T given below, and $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_2$ the column vectors of A.

$$A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3] = \begin{bmatrix} 0 & 6 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}, \ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 9 \\ -3 \\ 1 \end{bmatrix}.$$

We consider four sets of vectors: $B = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, the standard basis of $\boldsymbol{R}^3$, $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are given above, $S = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$, the set of column vectors of A, and $S' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, the orthonormal basis of $\boldsymbol{R}^3$ to be constructed in (b). You may use the fact that B is actually a basis of $\boldsymbol{R}^3$ and $A\boldsymbol{v}_1 = 6\boldsymbol{v}_1$, $A\boldsymbol{v}_2 = 2\boldsymbol{v}_2$ and $A\boldsymbol{v}_3 = -2\boldsymbol{v}_3$. To give your answer, show work and give your reason.

(a) Show that $S = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ is a basis of $\boldsymbol{R}^3$. (10 pts)

Sol. Since B is a basis of $\boldsymbol{R}^3$, $\dim(\boldsymbol{R}^3) = 3$.

$$\det(A) = \begin{vmatrix} 0 & 6 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 4 \end{vmatrix} = -\begin{vmatrix} 6 & 0 \\ 2 & 4 \end{vmatrix} = -24 \neq 0.$$

Hence A is invertible. If $k_1\boldsymbol{a}_1 + k_2\boldsymbol{a}_2 + k_3\boldsymbol{a}_3 = \boldsymbol{0}$, then $A[k_1, k_2, k_3]^T = \boldsymbol{0}$ and $[k_1, k_2, k_3]^T = A^{-1}\boldsymbol{0} = \boldsymbol{0}$. Therefore $k_1 = k_2 = k_3 = 0$ and S is linearly independent. By Theorem 1 (a), $\mathrm{Span}(S) = \boldsymbol{R}^3$ as $\dim(\boldsymbol{R}^3) = 3$. Therefore S is a basis of $\boldsymbol{R}^3$. ■

One can show the linear independence of S just by solving a system of linear equations. It is not difficult either.

(b) Using the basis S, find an orthonormal basis $S' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ of $\boldsymbol{R}^3$ with respect to the usual Euclidean inner product by the Gram-Schmidt process. Note that $\boldsymbol{u}_1 = \boldsymbol{a}_1 = \boldsymbol{e}_2$. (10 pts)

Sol. $\boldsymbol{u}_1 = \boldsymbol{e}_2$. Since $\langle \boldsymbol{a}_2, \boldsymbol{u}_1 \rangle = 2$,

$$\boldsymbol{u}_2' = \boldsymbol{a}_2 - \langle \boldsymbol{a}_2, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 = \boldsymbol{a}_2 - 2\boldsymbol{u}_1 = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}.$$

Since $\|\boldsymbol{u}_2'\|^2 = 40$,

$$\boldsymbol{u}_2 = \frac{1}{\|\boldsymbol{u}_2\|}\boldsymbol{u}_2' = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Now $\langle \boldsymbol{a}_3, \boldsymbol{u}_1 \rangle = 3$, $\langle \boldsymbol{a}_3, \boldsymbol{u}_2 \rangle = 4/\sqrt{10}$,

$$\boldsymbol{u}_3' = \boldsymbol{a}_3 - 3\boldsymbol{u}_1 - \frac{4}{\sqrt{10}}\boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \frac{6}{5}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}.$$

Therefore $\|\boldsymbol{u}_3'\| = 6\sqrt{10}/5$ and

$$\boldsymbol{u}_3 = \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \text{ and } S' = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}\right\}.$$

■

(c) Express each of $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ as a linear combination of the orthonormal basis S'. (5 pts)

Sol. Since S' is an orthonormal basis, for $\boldsymbol{v} \in V$,

$$\boldsymbol{v} = \langle \boldsymbol{v}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \langle \boldsymbol{v}, \boldsymbol{u}_2 \rangle \boldsymbol{u}_2 + \langle \boldsymbol{v}, \boldsymbol{u}_3 \rangle \boldsymbol{u}_3.$$

As for $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$, the computation is easy and we get

$$\boldsymbol{e}_1 = \frac{3}{\sqrt{10}}\boldsymbol{u}_2 - \frac{1}{\sqrt{10}}\boldsymbol{u}_3, \ \boldsymbol{e}_3 = \frac{1}{\sqrt{10}}\boldsymbol{u}_2 + \frac{3}{\sqrt{10}}\boldsymbol{u}_3.$$

■

(d) Find $[I]_{S',B}$ and $[I]_{B,S'}$, where $I : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ ($\boldsymbol{x} \mapsto \boldsymbol{x}$) is the identity operator on $\boldsymbol{R}^3$. (10 pts)

Sol.

$$[I]_{S',B} = [[\boldsymbol{e}_1]_{S'}, [\boldsymbol{e}_2]_{S'}, [\boldsymbol{e}_3]_{S'}] = \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{bmatrix}, \text{ and}$$

$$[I]_{B,S} = [[\boldsymbol{u}_1]_B, [\boldsymbol{u}_2]_B, [\boldsymbol{u}_3]_B] = \begin{bmatrix} 0 & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}.$$

■

(e) Express $[T]_{S'}$ using A, $[I]_{S',B}$ and $[I]_{B,S'}$. (5 pts)

Sol.

$$[T]_{S'} = [T]_{S',S'} = [I]_{S',B}[T]_{B,B}[I]_{B,S'} = [I]_{S',B}A[I]_{B,S'}.$$

■

(f) Find the matrix $[T]_{B'}$ for T with respect to the basis $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. (10 pts)

Sol. Since $A\boldsymbol{v}_1 = 6\boldsymbol{v}_1$, $A\boldsymbol{v}_2 = 2\boldsymbol{v}_2$ and $A\boldsymbol{v}_3 = -2\boldsymbol{v}_3$,

$$\begin{aligned} [T]_{B'} &= [[T(\boldsymbol{v}_1)]_{B'}, [T(\boldsymbol{v}_2)]_{B'}, [T(\boldsymbol{v}_3)]_{B'}] = [[6\boldsymbol{v}_1]_{B'}, [2\boldsymbol{v}_2]_{B'}, [-2\boldsymbol{v}_3]_{B'}] \\ &= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \end{aligned}$$

■

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