3 Vector Spaces and Subspaces

3.1 Definition of Vector Spaces

In the following K denotes either the real number field $\boldsymbol{R}$, the set of real numbers with two binary operations, i.e., addition and multiplication, or the complex number field $\boldsymbol{C}$. K can be replaced by any algebraic structure called a *field* but assume $K = \boldsymbol{R}$ unless otherwise stated. Elements of K are called scalars.

$K = \{0, 1\}$ with addition and multiplication defined by $0+0 = 0$, $0+1 = 1+0 = 1$, $1+1 = 0$, and $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$ is another example of a field.

Definition 3.1 [**Vector Space Axioms**] Let (K be a field and let) V be a set on which two operations are defined: additions and multiplication by scalars (numbers). (By *addition* we mean a rule for associating with each pair of elements $\boldsymbol{u}, \boldsymbol{v} \in V$ an element $\boldsymbol{u} + \boldsymbol{v} \in V$, called the *sum* of $\boldsymbol{u}$ and $\boldsymbol{v}$, by *scalar multiplication* we mean a rule for associating with each scalar k and each element $\boldsymbol{u} \in V$ an element $k\boldsymbol{u} \in V$, called the *scalar multiple* of $\boldsymbol{u}$ by k.) If the following axioms are satisfied, then we call V a *vector space* (over K) and we call the elements in V *vectors*.

1. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are elements in V, then $\boldsymbol{u} + \boldsymbol{v}$ is in V.
2. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$.
3. $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$.
4. There is an element $\mathbf{0} \in V$, called a *zero vector* for V, such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$ for all $\boldsymbol{u} \in V$.
5. For each $\boldsymbol{u} \in V$, there is an element $-\boldsymbol{u} \in V$, called a *negative* of $\boldsymbol{u}$, such that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$.
6. If k is a scalar and $\boldsymbol{u}$ is an element in V, then $k\boldsymbol{u}$ is in V.
7. $k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and any scalar k.
8. $(k + m)\boldsymbol{u} = k\boldsymbol{u} + m\boldsymbol{u}$ for any vector $\boldsymbol{u} \in V$ and all scalars k and m.
9. $k(m\boldsymbol{u}) = (km)\boldsymbol{u}$ for any vector $\boldsymbol{u} \in V$ and all scalars k and m.
10. $1\boldsymbol{u} = \boldsymbol{u}$ for any vector $\boldsymbol{u} \in V$.

Vector spaces over $\boldsymbol{R}$ are called *real vector spaces* and vector spaces over $\boldsymbol{C}$ *complex vector spaces.*

Remarks.

1. The zero element in Definition 3.1 4 is unique, i.e., if $\mathbf{0}'$ is another element in V satisfying $\boldsymbol{u} + \mathbf{0}' = \boldsymbol{u}$ for all $\boldsymbol{u} \in V$, then $\mathbf{0} = \mathbf{0}'$. See the following.

$$\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}' + \mathbf{0} = \mathbf{0}'.$$

2. The negative of $\boldsymbol{u}$ is unique for each $\boldsymbol{u} \in V$ in Definition 3.1 5, i.e., if $(-\boldsymbol{u})'$ is another element in V satisfying $\boldsymbol{u} + (-\boldsymbol{u})' = \mathbf{0}$, then $-\boldsymbol{u} = (-\boldsymbol{u})'$.

Proposition 3.1 (5.1.1) *Let V be a vector space, $\boldsymbol{u}$ a vector in V, and k a scalar; then:*

(a) $0\boldsymbol{u} = \mathbf{0}$.

(b) $k\mathbf{0} = \mathbf{0}$.

(c) $(-1)\boldsymbol{u} = -\boldsymbol{u}$.

(d) *If $k\boldsymbol{u} = \mathbf{0}$, then $k = 0$ or $\boldsymbol{u} = \mathbf{0}$.*

Proof. See page 226 for (a) and (c). ■

Example 3.1 [**Examples of Vector Spaces**]

1. The set $V = \boldsymbol{R}^n$ with the standard operations of addition and scalar multiplication is a (real) vector space for every positive integer n. R, R^2, R^3 are three important special cases.

2. For positive integers m, n let $M_{m,n}(= M_{m,n}(\boldsymbol{R}))$ denotes the set of all $m \times n$ matrices with real entries. Then $V = M_{m,n}$ becomes a (real) vector space with the operations of matrix addition and scalar multiplication.

3. Let X be a set and $F(X, \boldsymbol{R})$ the set of real-valued functions defined on X. For $f \in F(X, \boldsymbol{R})$, $f(x)$ denotes the value of f at $x \in X$. Then $V = F(X, \boldsymbol{R})$ becomes a (real) vector space with respect to the operations defined by the following.

$$(f + g)(x) = f(x) + g(x),\ (kf)(x) = kf(x) \text{ for all } f, g \in V \text{ and } k \in \boldsymbol{R}.$$

3.2 Subspaces

Definition 3.2 A subset W of a vector space V is called a *subspace* of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem 3.2 (5.2.1) *If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.*

(a) *$\boldsymbol{u} + \boldsymbol{v} \in W$ for all $\boldsymbol{u}, \boldsymbol{v} \in W$.*

(b) *$k\boldsymbol{u} \in W$ for all $\boldsymbol{u} \in W$ and all scalars k.*

Proof. See page 230. We apply Proposition 3.1 (c). ■

Proposition 3.3 (5.2.2) *Let A be an $m \times n$ matrix, and $T = T_A$ a linear transformation defined by*

$$T : \boldsymbol{R}^n \to \boldsymbol{R}^m \ (\boldsymbol{x} \mapsto A\boldsymbol{x}).$$

Then $W = \{\boldsymbol{v} \in \boldsymbol{R}^n \mid T(\boldsymbol{v}) = \boldsymbol{0}\}$ is a subspace of a vector space $V = \boldsymbol{R}^n$. W is called the kernel of the linear transformation T and is denoted by $\mathrm{Ker}(T)$.

Proof. See page 233. ■

Example 3.2 Let $V = \boldsymbol{R}^3$. Then the plane W through the origin in $\boldsymbol{R}^3$ defined below is a subspace of V:

$$W = \{(x, y, z)^T \in \boldsymbol{R}^3 \mid ax + by + cz = 0, \ \text{ where } a, b, c \in \boldsymbol{R}\}.$$

Let $A = [a, b, c] \in M_{1,3}$. Then $W = \mathrm{Ker}(T_A)$. Hence W is a subspace of V by Proporition 3.3. In particular W is a vector space.

Definition 3.3 [**Linear Combination**] A vector $\boldsymbol{w}$ is called a *linear combination* of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ if it can be expressed in the form

$$\boldsymbol{w} = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

Theorem 3.4 (5.2.3) *If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ are vectors in a vector space V, then*

(a) *The set W of all linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ is a subspace of V.*

(b) *W is the smallest subspace of V that contains $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ in the sense that every other subspace of V that contains $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ must contain W.*

Proof. See page 236. ■

Definition 3.4 If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ is a set of vectors in a vector space V, then the subspace W of V consisting of all linear combinations of the vectors in S is called the *space spanned* by $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$, and we say that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ *span* W. To indicate that W is the space spanned by the vectors in the set $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$, we write

$$W = \mathrm{Span}(S) \ \text{ or } \ W = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}.$$

Exercise 3.1 [Quiz 3]

1. Let V be a vector space and k a scalar. Show $k\boldsymbol{0} = \boldsymbol{0}$. In each step of your proof quote the axiom applied. [Hint: Exercise 5.1.29]

2. Let A, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be as follows.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix}, \ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -8 \\ -6 \end{bmatrix}.$$

   (a) Let $B = (A - I)^2$. Show that $W = \{\boldsymbol{v} \in \boldsymbol{R}^3 \mid B\boldsymbol{v} = 10\boldsymbol{v}\}$ is a subspace of $V = \boldsymbol{R}^3$.

   (b) Determine whether or not $\boldsymbol{v}_3$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.