4 Linear Independence and Basis

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \boldsymbol{R}^3$ given below and $W = \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -8 \\ -6 \end{bmatrix}.$$

Since $\boldsymbol{v}_3 = 5\boldsymbol{v}_1 + \boldsymbol{v}_2$, $W = \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \{a\boldsymbol{v}_1 + b\boldsymbol{v}_2 \mid a, b \in K\}$. Moreover, the expression $a\boldsymbol{v}_1 + b\boldsymbol{v}_2$ is unique, i.e., if $a\boldsymbol{v}_1 + \boldsymbol{v}_2 = a'\boldsymbol{v}_1 + b'\boldsymbol{v}_2$ then $a = a'$ and $b = b'$. Hence $f : W \to \boldsymbol{R}^2$ $(a\boldsymbol{v}_1 + b\boldsymbol{v}_2 \mapsto (a, b))$ is a bijection, and W can be regarded as a vector space similar to $\boldsymbol{R}^2$. We call $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ a basis of W and $\dim(W) = 2$. As for $\boldsymbol{R}^2$, let $\boldsymbol{e}_1 = (1, 0)$ and $\boldsymbol{e}_2 = (0, 1)$. Then $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a (standard) basis, and $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ in W plays a similar role of $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ in $\boldsymbol{R}^2$. In this case W is a plane through the origin in $\boldsymbol{R}^3$, and it is natural to think W as an object similar to $\boldsymbol{R}^2$. In the following we study a basis and dimension of a general vector space.

4.1 Linear Independence

Definition 4.1 Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ be a nonempty set of vectors. If the equation

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = \boldsymbol{0}$$

has only one solution, namely, $k_1 = k_2 = \cdots = k_r = 0$, then S is called a *linearly independent* set. If there are other solutions, then S is called a *linearly dependent* set.

Proposition 4.1 (5.3.1, 5.4.1) *Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ be a nonempty set of vectors. Then the following are equivalent.*

(a) *S is a linearly independent set.*

(b) *No vector in S is expressible as a linear combination of the other vectors in S.*

(c) *For each vector $\boldsymbol{v}$, $k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = \boldsymbol{v}$ has at most one solution, i.e., if*

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = k_1'\boldsymbol{v}_1 + k_2'\boldsymbol{v}_2 + \cdots + k_r'\boldsymbol{v}_r$$

then $k_1 = k_1'$, $k_2 = k_2', \ldots, k_r = k_r'$.

Proof. (a)$\Rightarrow$(c): Suppose

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = k_1'\boldsymbol{v}_1 + k_2'\boldsymbol{v}_2 + \cdots + k_r'\boldsymbol{v}_r.$$

Then

$$(k_1 - k_1')\boldsymbol{v}_1 + (k_2 - k_2')\boldsymbol{v}_2 + \cdots + (k_r - k_r')\boldsymbol{v}_r = \boldsymbol{0}.$$

Now (a) implies that $k_1 - k_1' = k_2 - k_2' = \cdots = k_r - k_r' = 0$.

(c)⇒(b): Suppose not. Then

$$\boldsymbol{v}_i = k_1\boldsymbol{v}_1 + \cdots + k_{i-1}\boldsymbol{v}_{i-1} + k_{i+1}\boldsymbol{v}_{i+1} + \cdots + k_r\boldsymbol{v}_r.$$

Comparing the coefficients of $\boldsymbol{v}_i$ on both hand sides, we have $1 = 0$ by (c), a contradiction.

(b)⇒(a): Suppose

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = \boldsymbol{0}$$

such that not all $k_1, k_2, \ldots, k_r$ zero. Say $k_i \neq 0$. Then

$$\boldsymbol{v}_i = \frac{-k_1}{k_i}\boldsymbol{v}_1 + \cdots + \frac{-k_{i-1}}{k_i}\boldsymbol{v}_{i-1} + \frac{-k_{i+1}}{k_i}\boldsymbol{v}_{i+1} + \cdots + \frac{-k_r}{k_i}\boldsymbol{v}_r.$$

This contradicts (b). ■

Theorem 4.2 (1.2.1) *A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.*

Theorem 4.3 (5.3.3) *Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ be a set of vectors in $\boldsymbol{R}^n$. If $r > n$, then S is linearly dependent. In particular, if $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r]$, then a system of linear equation $A\boldsymbol{x} = \boldsymbol{0}$ has a nonzero solution.*

Proof. See Theorem 1.2.1. ■

Proposition 4.4 (5.3.4) *If the functions $\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n$ have $n-1$ continuous derivatives on the interval (a, b), and if the Wronskian of these functions is not identically zero on (a, b), then these functions form a linearly independent vectors in $C^{(n-1)}(a, b)$.*

4.2 Basis and Dimension

Definition 4.2 If V is a vector space and $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a set of vectors fo V, then S is called a *basis* for V if the following two conditions hold:

(a) S is linearly independent.

(b) S spans V, i.e., every vector in V can be written as a linear combination of vectors in S.

V is called *finite-dimensional* if it contains a finite set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ that forms a basis. If no such set exists, V is called *infinite-dimensional*.

Theorem 4.5 (5.4.2) *Let V be a finite-dimensional vector space, and let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a basis.*

(a) *If a set has more than n vectors, then it is linearly dependent.*

(b) *If a set has fewer than n vectors, then it does not span V.*

Proof. (a): Since $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m \in V = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$, there exist $a_{i,j}$ $(1 \le i \le m,\ 1 \le j \le n)$ such that

$$\begin{aligned}
\boldsymbol{w}_1 &= a_{1,1}\boldsymbol{v}_1 + a_{1,2}\boldsymbol{v}_2 + \cdots + a_{1,n}\boldsymbol{v}_n \\
\boldsymbol{w}_2 &= a_{2,1}\boldsymbol{v}_1 + a_{2,2}\boldsymbol{v}_2 + \cdots + a_{2,n}\boldsymbol{v}_n \\
\vdots\ &\quad \vdots \\
\boldsymbol{w}_m &= a_{m,1}\boldsymbol{v}_1 + a_{m,2}\boldsymbol{v}_2 + \cdots + a_{m,n}\boldsymbol{v}_n.
\end{aligned}$$

Suppose $m > n$. Then by Theorem 4.3, there exist scalars $k_1, k_2, \ldots, k_m$ not all zero such that

$$k_1 \begin{bmatrix} a_{1,1} \\ a_{1.2} \\ \vdots \\ a_{1,n} \end{bmatrix} + k_2 \begin{bmatrix} a_{2,1} \\ a_{2.2} \\ \vdots \\ a_{2,n} \end{bmatrix} + \cdots + k_m \begin{bmatrix} a_{m,1} \\ a_{m.2} \\ \vdots \\ a_{m,n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then

$$\begin{aligned}
&k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2 + \cdots + k_m\boldsymbol{w}_m \\
&= \sum_{j=1}^{m} k_j(a_{j,1}\boldsymbol{v}_1 + a_{j,2}\boldsymbol{v}_2 + \cdots + a_{j,n}\boldsymbol{v}_n) = \sum_{j=1}^{m}\sum_{i=1}^{n} k_j a_{j,i}\boldsymbol{v}_i \\
&= \sum_{i=1}^{n}\Big(\sum_{j=1}^{m} k_j a_{j,i}\Big)\boldsymbol{v}_i = \sum_{i=1}^{n}(k_1a_{1,i} + k_2a_{2,i} + \cdots + k_m a_{m,i})\boldsymbol{v}_i \\
&= \mathbf{0}
\end{aligned}$$

Therefore $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m\}$ is a linearly dependent set.

(b): Suppose $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m\}$ spans V and $m < n$. Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \in V$, there exist $a_{i,j}$ $(1 \le i \le m,\ 1 \le j \le n)$ such that

$$\begin{aligned}
\boldsymbol{v}_1 &= a_{1,1}\boldsymbol{w}_1 + a_{2,1}\boldsymbol{w}_2 + \cdots + a_{m,1}\boldsymbol{w}_m \\
\boldsymbol{v}_2 &= a_{1,2}\boldsymbol{w}_1 + a_{2,2}\boldsymbol{w}_2 + \cdots + a_{m,2}\boldsymbol{w}_m \\
\vdots\ &\quad \vdots \\
\boldsymbol{v}_n &= a_{1,n}\boldsymbol{w}_1 + a_{2,n}\boldsymbol{w}_2 + \cdots + a_{m,n}\boldsymbol{w}_m.
\end{aligned}$$

Since $n > m$, by by Theorem 4.3, there exist scalars $k_1, k_2, \ldots, k_n$ not all zero such that

$$k_1 \begin{bmatrix} a_{1,1} \\ a_{2.1} \\ \vdots \\ a_{m,1} \end{bmatrix} + k_2 \begin{bmatrix} a_{1,2} \\ a_{2.2} \\ \vdots \\ a_{m,2} \end{bmatrix} + \cdots + k_n \begin{bmatrix} a_{1,n} \\ a_{2.n} \\ \vdots \\ a_{m,n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_n\boldsymbol{v}_n$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} k_j(a_{1,j}\boldsymbol{w}_1 + a_{2,j}\boldsymbol{w}_2 + \cdots + a_{m,j}\boldsymbol{w}_n) = \sum_{j=1}^{n}\sum_{i=1}^{m} k_j a_{i,j}\boldsymbol{w}_i \\
&= \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} k_j a_{i,j}\Big)\boldsymbol{w}_i = \sum_{i=1}^{m}(k_1 a_{i,1} + k_2 a_{i,2} + \cdots + k_m a_{i,m})\boldsymbol{v}_i \\
&= \boldsymbol{0}
\end{aligned}
$$

This contradicts that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is a linearly independent set. ■

Corollary 4.6 (5.4.3) *All bases for a finite-dimensional vector space have the same number of vectors.*

Definition 4.3 The *dimension* of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V. (In addition, we define the zero vector space to have dimension zero.)

Proposition 4.7 (5.4.4) *Let S be a nonempty set of vectors in a vector space V.*

(a) *If S is a linearly independent set, and $\boldsymbol{v} \notin \mathrm{Span}(S)$, then $S \cup \{\boldsymbol{v}\}$ is a linearly independent set.*

(b) *If $\boldsymbol{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, then $\mathrm{Span}(S \setminus \{\boldsymbol{v}\}) = \mathrm{Span}(S)$.*

Theorem 4.8 (5.4.5, 5.4.6, 5.4.7) *Let V be an n-dimensional vector space, and S a set of vectors in V*

(a) *Suppose S has exactly n vectors. Then S is linearly independent if and only if S spans V.*

(b) *If S spans V but not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(c) *If S is linearly independent that is not already a basis for V, then S can be enlarged to a basis of V by inserting appropriate vectors into S.*

(d) *If W is a subspace of V, then $\dim(W) \le \dim(V)$. Moreover if $\dim(W) = \dim(V)$, then $W = V$.*

Exercise 4.1 [Quiz 4] Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ be vectors in $\boldsymbol{R}^3$ given below.

$$
\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -8 \\ -6 \end{bmatrix},\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
$$

1. Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis of $U = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.
2. Show that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a basis of $\boldsymbol{R}^3$.
3. Show that $\boldsymbol{e}_1 \notin \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.
4. Show that $\{\boldsymbol{e}_1, \boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis of $\boldsymbol{R}^3$.
5. Express $\boldsymbol{e}_2$ as a linear combination of $\boldsymbol{e}_1, \boldsymbol{v}_1, \boldsymbol{v}_2$.