5 Dimensions of Subspaces

5.1 Row Space, Column Space and Nullspace

Definition 5.1 For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ & & \cdots & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix},$$

the vectors

$$\begin{aligned} \boldsymbol{r}_1 &= [a_{1,1}, a_{1,2}, \ldots, a_{1,n}] \\ \boldsymbol{r}_2 &= [a_{2,1}, a_{2,2}, \ldots, a_{2,n}] \\ &\vdots \\ \boldsymbol{r}_m &= [a_{m,1}, a_{m,2}, \ldots, a_{m,n}] \end{aligned}$$

in $\boldsymbol{R}^n$ formed from the rows of A are called the *row vectors* of A and the vectors

$$\boldsymbol{c}_1 = \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix}, \boldsymbol{c}_2 = \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{bmatrix}, \cdots, \boldsymbol{c}_n = \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix}$$

in $\boldsymbol{R}^n$ formed from the columns of A are called the *column vectors* of A.

Let $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T$ and $\boldsymbol{y} = [y_1, y_2, \ldots, y_m]$. Then the following are useful.

$$\begin{aligned} A\boldsymbol{x} &= [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n, \\ \boldsymbol{y}A &= [y_1, y_2, \ldots, y_m] \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_m \end{bmatrix} = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + \cdots + y_m\boldsymbol{r}_m. \end{aligned}$$

Definition 5.2 Let A be an $m \times n$ matrix, then the subspace of $\boldsymbol{R}^n$ spanned by the row vectors of A is called the *row space* of A, and the subspace of $\boldsymbol{R}^m$ spanned by the column vectors of A is called the *column* space of A. The solution space of the homogeneous system of equations $A\boldsymbol{x} = \boldsymbol{0}$, which is a subspace of $\boldsymbol{R}^n$, is called the *nullspace* of A.

The dimension of the column space of a matrix A is called the *rank* of A and is denoted by rank(A). The dimension of the nullspace of A is called the *nullity* of A and is denoted by nullity(A).

Let A be an $m \times n$ matrix and $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m$ and $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$ vectors defined above. Then

Row Space of A: $\mathcal{R}(A) = \mathrm{Span}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m\} \subset \boldsymbol{R}^n$.

Column Space of A: $\mathcal{C}(A) = \mathrm{Span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n\} \subset \boldsymbol{R}^m$. $\mathrm{rank}(A) = \dim(\mathcal{C}(A))$.

Nullspace of A: $\mathcal{N}(A) = \{\boldsymbol{v} \in \boldsymbol{R}^n \mid A\boldsymbol{v} = \boldsymbol{0}\} = \mathrm{Ker}(T_A) \subset \boldsymbol{R}^n$, where T_A is a linear transformation define by $T_A : \boldsymbol{R}^n \to \boldsymbol{R}^m$ $(\boldsymbol{x} \mapsto A\boldsymbol{x})$. $\mathrm{nullity}(A) = \dim(\mathcal{N}(A))$.

These are all subspaces. Namely $\mathcal{R}(A)$ is a subspace of $\boldsymbol{R}^n$ (row vector space), $\mathcal{C}(A)$ a subspace of $\boldsymbol{R}^m$ (column vector space), and $\mathcal{N}(A)$ a subspace of $\boldsymbol{R}^n$ (column vector space). See Proposition 3.3 and Theorem 3.4.

Proposition 5.1 (5.5.1) *A system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ is in the column space of A.*

Proof. Let $A = [\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n]$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$. Then

$$A\boldsymbol{x} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n.$$

Hence $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b} \in \mathrm{Span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n\} = \mathcal{C}(A)$. ∎

Theorem 5.2 (5.5.2) *If $\boldsymbol{x}_0$ denotes any single solution of a consistent linear system $A\boldsymbol{x} = \boldsymbol{b}$, and if $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ form a basis for the nullspace of A, then every solution of $A\boldsymbol{x} = \boldsymbol{b}$ can be expressed in the form*

$$\boldsymbol{x} = \boldsymbol{x}_0 + c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$$

and, conversely, for all choices of scalars $c_1, c_2, \cdots, c_k$, the vector $\boldsymbol{x}$ in this formula is a solution of $A\boldsymbol{x} = \boldsymbol{b}$.

Proof. By assumption, $A\boldsymbol{x}_0 = \boldsymbol{b}$ and $A\boldsymbol{x} = \boldsymbol{b}$. Then

$$A(\boldsymbol{x} - \boldsymbol{x}_0) = A\boldsymbol{x} - A\boldsymbol{x}_0 = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0}.$$

Hence $\boldsymbol{x} - \boldsymbol{x}_0 \in \mathcal{N}(A)$. Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is a basis for the null space $\mathcal{N}(A)$ of A, there exist scalars $c_1, c_2, \ldots, c_k$ such that

$$\boldsymbol{x} - \boldsymbol{x}_0 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k.$$

Therefore

$$\boldsymbol{x} = \boldsymbol{x}_0 + c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k.$$

On the other hand, let $\boldsymbol{x} = \boldsymbol{x}_0 + c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$. Then

$$A\boldsymbol{x} = A(\boldsymbol{x}_0 + c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k) = A\boldsymbol{x}_0 + c_1A\boldsymbol{v}_1 + c_2A\boldsymbol{v}_2 + \cdots + c_kA\boldsymbol{v}_k = \boldsymbol{b}.$$

Recall that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathcal{N}(A) = \{\boldsymbol{v} \in \boldsymbol{R}^n \mid A\boldsymbol{v} = \boldsymbol{0}\}$. Hence $\boldsymbol{x}_0 + c_1\boldsymbol{v}_1 + c_1\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$ is a solution of $A\boldsymbol{x} = \boldsymbol{b}$. ∎

Remarks.

1. $\boldsymbol{x}_0$ in the previous theorem is called a *particular sotution* of $A\boldsymbol{x} = \boldsymbol{b}$.
2. $\boldsymbol{x}_0 + c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$ is called the *general solution* of $A\boldsymbol{x} = \boldsymbol{b}$.
3. $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$ is called the *general solution* of $A\boldsymbol{x} = \underline{0}$.
4. If a particular solution x_0 and a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ of the nullspace of A, i.e., $\mathcal{N}(A)$ are given, each solution $A\boldsymbol{x} = \boldsymbol{b}$ is expressed uniquely in the form

$$\boldsymbol{x} = \boldsymbol{x}_0 + c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k,$$

   i.e., scalars $c_1, c_2, \ldots, c_k$ are uniquely determined for each solution $\boldsymbol{x}$.

Lemma 5.3 *Let A be an $m \times r$ matrix and B an $r \times n$ matrix. Then*

$$\mathcal{R}(AB) \subset \mathcal{R}(B),\ \mathcal{C}(AB) \subset \mathcal{C}(A).$$

Proof. Let $A = [a_{i,j}] = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_r]$ and $B = [b_{i,j}]$, $B^T = [\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_r]$. Then the i-th row of AB is

$$a_{i,1}\boldsymbol{b}_1^T + a_{i,2}\boldsymbol{b}_2^T + \cdots + a_{1,r}\boldsymbol{b}_r^T \in \mathcal{R}(B),$$

and the j-th column of AB is

$$b_{1,j}\boldsymbol{a}_1 + b_{2,j}\boldsymbol{a}_2 + \cdots + b_{r,j}\boldsymbol{a}_r \in \mathcal{C}(A).$$

■

Proposition 5.4 (5.5.3, 5.5.4, 5.5.5) *Let A be an $m \times n$ matrix and P an invertible matrix of size $m \times m$,*

(a) *Elementary row operations do not change the nullspace of a matrix. Moreover, $\mathcal{N}(A) = \mathcal{N}(PA)$.*

(b) *Elementary row operations do not change the row space of a matrix. Moreover, $\mathcal{R}(A) = \mathcal{R}(PA)$.*

(c) *$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\} \subset \boldsymbol{R}^n$ is a linearly independent set if and only if $\{P\boldsymbol{v}_1, P\boldsymbol{v}_2, \ldots, P\boldsymbol{v}_r\} \subset \boldsymbol{R}^n$ is a linearly independent set.*

Proof. (a): $\boldsymbol{v} \in \mathcal{N}(A) \Leftrightarrow A\boldsymbol{v} = \boldsymbol{0} \Leftrightarrow PA\boldsymbol{v} = \boldsymbol{0} \Leftrightarrow \boldsymbol{v} \in \mathcal{N}(PA)$. Note that if $PA\boldsymbol{v} = \boldsymbol{0}$, then $A\boldsymbol{v} = P^{-1}PA\boldsymbol{v} = P^{-1}\boldsymbol{0} = \boldsymbol{0}$.

(b): Lemma 5.3 is applicable. Let $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_m$ be the rows of P.Then $\mathcal{R}(PA) = \text{Span}\{\boldsymbol{p}_1 A, \boldsymbol{p}_2 A, \ldots, \boldsymbol{p}_m A\} \subset \mathcal{R}(A)$. Similarly we have $\mathcal{R}(A) = \mathcal{R}(P^{-1}PA) \subset \mathcal{R}(PA)$. Hence $\mathcal{R}(A) = \mathcal{R}(PA)$.

(c): Since $k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = \boldsymbol{0}$ if and only if $k_1P\boldsymbol{v}_1 + k_2P\boldsymbol{v}_2 + \cdots + k_rP\boldsymbol{v}_r = \boldsymbol{0}$, the assertion is clear. ■

5.2 Rank and Nullity

Proposition 5.5 (5.5.6) *If a matrix R is in row-echeron form, then the row vectors with the leading 1's form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.*

Theorem 5.6 (5.6.1) *If A is any matrix, then the row space and column space of A have the same dimension. Hence* $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

Proof. Let A' be the reduced row-echelon form of A. Then

$$\mathrm{rank}(A) = \mathrm{rank}(A') = \dim(\mathcal{C}(A')) = \dim(\mathcal{R}(A')) = \dim(\mathcal{R}(A)) = \mathrm{rank}(A^T).$$

■

Theorem 5.7 ((5.6.3) Dimension Theorem for Matrices) *If A is a matrix with n columns, then*

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = n.$$

Proof. Let B be the reduced row-echelon form of A. Then $\mathrm{rank}(A) = \mathrm{rank}(B)$ and $\mathrm{nullity}(A) = \mathrm{nullity}(B)$. Hence it suffices to prove the assertion for a matrix which is already in a reduced row-echelon form. Then $\mathrm{rank}(A)$ is the number of leading 1s, and $\mathrm{nullity}(A)$ is the number of free variables. ■

Example 5.1 Let us consider the following system of linear equations and the augmented matrix $[A, \boldsymbol{b}]$.

$$\begin{cases} x_1 + 0x_2 + x_3 + 0x_4 + x_5 + 3x_6 &= -1 \\ -x_1 + 0x_2 - x_3 + 0x_4 + 0x_5 - 4x_6 &= -1 \\ 0x_1 + x_2 - 2x_3 + 3x_4 + 0x_5 - x_6 &= 3 \\ -2x_1 - 2x_2 + 2x_3 - 6x_4 - 2x_5 - 4x_6 &= -4 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ -2 & -2 & 2 & -6 & -2 & -4 & -4 \end{bmatrix}$$

$$[A, \boldsymbol{b}] \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [A', \boldsymbol{b}']$$

Then there is an invertible matrix P such that $PA = A'$ and $P\boldsymbol{b} = \boldsymbol{b}'$. Let $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_6$ be the column vectors of A, $\boldsymbol{c}'_1, \boldsymbol{c}'_2, \ldots, \boldsymbol{c}'_6$ the column vectors of A', $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4$ be the row vectors of A, $\boldsymbol{r}'_1, \boldsymbol{r}'_2, \boldsymbol{r}'_3, \boldsymbol{r}'_4$ the row vectors of A'.

1. $A\boldsymbol{x} = \boldsymbol{b} \Leftrightarrow A'\boldsymbol{x} = \boldsymbol{b}'$. Hence the solution to the first equation is the solution to the second and vice versa.

2. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $A'\boldsymbol{x} = \boldsymbol{b}'$ is consistent.

3. $\mathcal{C}(A') = \mathrm{Span}\{\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_5\} = \{(a, b, c, 0)^T \mid a, b, c \in \boldsymbol{R}\}$. In particular $\boldsymbol{b}' \in \mathcal{C}(A')$ and the equation is consistent.

4. $\mathcal{C}(A) = \text{Span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_5\}$. In particular the equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b} \in \text{Span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_5\}$. $\dim(\mathcal{C}(A)) = \dim(\mathcal{C}(A')) = 3$.

5. $\mathcal{R}(A) = \mathcal{R}(A') = \text{Span}\{\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_3\}$ and $\dim(\mathcal{R}(A)) = \dim(\mathcal{R}(A')) = 3$.

6. $\mathcal{N}(A) = \mathcal{N}(A') = \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ where $\boldsymbol{v}_1 = (-1, 2, 1, 0, 0, 0)^T$, $\boldsymbol{v}_2 = (0, -3, 0, 1, 0, 0)^T$, and $\boldsymbol{v}_3 = (-4, 1, 0, 0, 1, 1)^T$, and $\dim(\mathcal{N}(A)) = \dim(\mathcal{N}(A')) = 3$.

7. $\boldsymbol{x}_0 = (1, 3, 0, 0, -2, 0)^T$.

$$\begin{cases} x_1 &= 1 - s - 4u, \\ x_2 &= 3 + 2s - 3t + u, \\ x_3 &= s, \\ x_4 &= t, \\ x_5 &= -2 + u, \\ x_6 &= u. \end{cases} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ -2 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \cdot \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

s, t and u are parameters.

Exercise 5.1 [Quiz 5] Let A be the coefficient matrix, and B the augmented matrix of a system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{x} = [x_1, x_2, x_3, x_4, x_5, x_6]^T$. Let C be a reduced row-echelon form obtained from B by a series of elementary row operations.

$$B = [A, \boldsymbol{b}] = \begin{bmatrix} 3 & -3 & 0 & 3 & -6 & -3 & -6 \\ 2 & -2 & 1 & 5 & -3 & -1 & 1 \\ -3 & 3 & 0 & -3 & 6 & 1 & -8 \\ -1 & 1 & 2 & 5 & 4 & 0 & -9 \end{bmatrix} \to C = \begin{bmatrix} 1 & -1 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 3 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank(A) and nullity(A).

2. Find a basis of the row space of A.

3. Find a basis of the column space of A.

4. Find a basis of the nullspace of A.

5. Find the general solution of the equation $A\boldsymbol{x} = \boldsymbol{b}$.