Linear Algebra II *February 20, 2007*

Practice Exam 2006/7

(Toal: 140pts)

In the following you may quote the following theorems, but when you use Theorem 1 clarify which item (a) - (d) is applied.

Theorem 1 *Let V be an n-dimensional vector space, and S a set of vectors in V.*

(a) *Suppose S has exactly n vectors. Then S is linearly independent if and only if S spans V.*

(b) *If S spans V but not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(c) *If S is linearly independent that is not already a basis for V, then S can be enlarged to a basis of V by inserting appropriate vectors into S.*

(d) *If W is a subspace of V, then $\dim(W) \leq \dim(V)$. Moreover if $\dim(W) = \dim(V)$, then $W = V$.*

Theorem 2 *If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in a real inner product space, then*

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq \|\boldsymbol{u}\| \|\boldsymbol{v}\|.$$

Equality holds if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent.

1. Prove the following proposition.

   Proposition. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ be a nonempty set of vectors. Then the following are equivalent.
   (a) S is a linearly independent set.
   (b) For each vector $\boldsymbol{v}$, $k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = \boldsymbol{v}$ has at most one solution, i.e., if

   $$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = k_1'\boldsymbol{v}_1 + k_2'\boldsymbol{v}_2 + \cdots + k_r'\boldsymbol{v}_r$$

   then $k_1 = k_1'$, $k_2 = k_2', \ldots, k_r = k_r'$.

2. Let $T : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be a linear transformation and $A = [T]$ the standard matrix of T given below. Let $N = \mathrm{Ker}(T)$, $C = \mathrm{Im}(T)$, and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be as follows.

   $$A = \begin{bmatrix} -1 & 1 & 0 \\ 10 & 0 & 5 \\ 0 & 8 & 4 \end{bmatrix}, \ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 10 \\ 16 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

   (Note that $C = R(T)$ in the textbook.)

   (a) Find a basis of N.

   (b) Find a basis of C consisting of column vectors of A.

(c) Find an orthogonal basis of C with respect to the usual Euclidean inner product.

(d) Show that $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis of $\boldsymbol{R}^3$.

(e) Determine whether or not $\boldsymbol{v}_3$ is in C.

(f) Let $B = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the standard basis of $\boldsymbol{R}^3$. Find $[I]_{S,B}$, where $I : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ $(\boldsymbol{x} \mapsto \boldsymbol{x})$ (the identity operator).

(g) Find $[T]_S$, the matrix for T with respect to the basis S.

3. Recall the definition of an inner product:

> An *inner product* on a real vector space V is a function that associates a real number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ with each pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in V in such a way that the following axioms are satisfied for all vectors $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ in V and all scalars k.
>
> (a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$ (Symmetry axiom)
>
> (b) $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{z} \rangle = \langle \boldsymbol{u}, \boldsymbol{z} \rangle + \langle \boldsymbol{v}, \boldsymbol{z} \rangle$ (Additive axiom)
>
> (c) $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ (Homogeneity axiom)
>
> (d)

(a) The condition (d) is missing in the definition of an inner product above. State it.

(b) Give an example of an inner product on a real vector space, which is different from the usual Euclidean Inner Product, and show that it satisfies the conditions above.

(c) In an inner product space V, show

$$d(\boldsymbol{u}, \boldsymbol{v}) \leq d(\boldsymbol{u}, \boldsymbol{w}) + d(\boldsymbol{w}, \boldsymbol{v}) \text{ for all } \boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w} \in V.$$

4. Let V be an n-dimensional vector space, W_1 and W_2 subspaces of V. Set $U = \{\boldsymbol{w}_1 + \boldsymbol{w}_2 \mid \boldsymbol{w}_1 \in W_1 \text{ and } \boldsymbol{w}_2 \in W_2\}$. ($U$ is often denoted by $W_1 + W_2$.)

(a) Show that $W = W_1 \cap W_2 = \{\boldsymbol{w} \mid \boldsymbol{w} \in W_1 \text{ and } \boldsymbol{w} \in W_2\}$ is a subspace of V.

(b) Show that U is a subspace of V.

(c) Show that there is a set of vectors

$$S = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_s, \boldsymbol{w}_{s+1}, \boldsymbol{w}_{s+2}, \dots, \boldsymbol{w}_{s+t}, \boldsymbol{w}_{s+t+1}, \boldsymbol{w}_{s+t+2}, \dots, \boldsymbol{w}_{s+t+r}\}$$

of V satisfying the following conditions.

i. $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_s\}$ is a basis of $W = W_1 \cap W_2$,

ii. $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_s, \boldsymbol{w}_{s+1}, \boldsymbol{w}_{s+2}, \dots, \boldsymbol{w}_{s+t}\}$ is a basis of W_1; and

iii. $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_s, \boldsymbol{w}_{s+t+1}, \boldsymbol{w}_{s+t+2}, \dots, \boldsymbol{w}_{s+t+r}\}$ is a basis of W_2.

(d) Show that $U = \text{Span}(S)$.

(e) Show that S is a basis of U.

Problem 4 proves that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Solutions to Practice Exam 2006/7

In the following you may quote the following theorems, but when you use Theorem 1 clarify which item (a) - (d) is applied.

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(a) *Suppose S has exactly n vectors. Then S is linearly independent if and only if S spans V.*

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(d) *If W is a subspace of V, then $\dim(W) \leq \dim(V)$. Moreover if $\dim(W) = \dim(V)$, then $W = V$.*

Theorem 2 *If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in a real inner product space, then*

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq \|\boldsymbol{u}\| \|\boldsymbol{v}\|.$$

Equality holds if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent.

1. Prove the following proposition.

   Proposition. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ be a nonempty set of vectors. Then the following are equivalent.
   (a) S is a linearly independent set.
   (b) For each vector $\boldsymbol{v}$, $k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = \boldsymbol{v}$ has at most one solution, i.e., if

   $$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = k_1'\boldsymbol{v}_1 + k_2'\boldsymbol{v}_2 + \cdots + k_r'\boldsymbol{v}_r$$

   then $k_1 = k_1'$, $k_2 = k_2', \ldots, k_r = k_r'$.

   Sol. (a)⇒(b): Suppose

   $$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = k_1'\boldsymbol{v}_1 + k_2'\boldsymbol{v}_2 + \cdots + k_r'\boldsymbol{v}_r$$

   Then by subtracting the right hand side from the left,

   $$(k_1 - k_1')\boldsymbol{v}_1 + (k_2 - k_2')\boldsymbol{v}_2 + \cdots + (k_r - k_r')\boldsymbol{v}_r = \boldsymbol{0}.$$

   By (a), $k_1 - k_1' = k_2 - k_2' = \cdots = k_r - k_r' = 0$. Hence $k_1 = k_1'$, $k_2 = k_2', \ldots, k_r = k_r'$.
   (b)⇒(a): Suppose

   $$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = \boldsymbol{0}.$$

   Then

   $$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r = 0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \cdots + 0\boldsymbol{v}_r$$

   as the both hand sides are zero. By (b), $k_1 = k_2 = \cdots = k_r = 0$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ is linearly independent. ■

2. Let $T : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be a linear transformation and $A = [T]$ the standard matrix of T given below. Let $N = \mathrm{Ker}(T)$, $C = \mathrm{Im}(T)$, and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be as follows.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 10 & 0 & 5 \\ 0 & 8 & 4 \end{bmatrix}, \ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 10 \\ 16 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

(Note that $C = R(T)$ in the textbook.)

(a) Find a basis of N.

Sol. Since

$$\mathbf{0} = T(\boldsymbol{x}) = A\boldsymbol{x} = \begin{bmatrix} -1 & 1 & 0 \\ 10 & 0 & 5 \\ 0 & 8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

by solving the equation we get $[x, y, z]^T = t[-1, -1, 2] = t\boldsymbol{v}_3$. Hence $\{\boldsymbol{v}_3\}$ is a basis. (Note that a set of one nonzero vector is always linearly independent. In addition, in this case all solutions above, i.e., the vectors in N can be written as a scalar multiple of $\boldsymbol{v}_3$, $\{\boldsymbol{v}_3\}$ is a basis. One can choose any other nonzero scalar multiple of $\boldsymbol{v}_3$ as a basis vector.) ■

(b) Find a basis of C consisting of column vectors of A.

Sol. By (a) $\mathrm{nullity}(T) = 1$ and $\mathrm{rank}(T) = 3 - \mathrm{nullity}(T) = 2$. And $\mathrm{Im}(T) = (C)(A)$. (If $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$, then by (a), $-\boldsymbol{a}_1 - \boldsymbol{a}_2 + 3\boldsymbol{a}_3 = \mathbf{0}$ and three column vectors are linearly dependent. So $\mathrm{rank}(T) \leq 2$.) Since $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is clearly linearly independent, it forms a basis of $\mathrm{Im}(T) = R(T) = \mathcal{C}(A)$. ■

(c) Find an orthogonal basis of C with respect to the usual Euclidean inner product.

Sol. Since $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is a basis of C, $\{\boldsymbol{a}_1, \boldsymbol{a}_2 - \frac{\langle \boldsymbol{a}_2, \boldsymbol{a}_1 \rangle}{\|\boldsymbol{a}_1\|^2}\boldsymbol{a}_1\}$ is an orthogonal basis, where

$$\boldsymbol{a}_2 - \frac{\langle \boldsymbol{a}_2, \boldsymbol{a}_1 \rangle}{\|\boldsymbol{a}_1\|^2}\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix} - \frac{1}{101}\begin{bmatrix} -1 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 100/101 \\ 10/101 \\ 8 \end{bmatrix}. \quad ■$$

(d) Show that $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis of $\boldsymbol{R}^3$.

Sol. Since $\dim(\boldsymbol{R}^3) = 3$, it suffices to show that S is linearly independent. Let $B = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$. If S is linearly dependent, $x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3 = \mathbf{0}$ has a nonzero solution. Hence $B\boldsymbol{x} = \mathbf{0}$ has a nonzero solution. But $\det(B) = 162 \neq 0$. Hence B is invertible. So $\boldsymbol{x} = \mathbf{0}$ and S is linearly independent. (One can show the same by solving the linear equation.) ■

(e) Determine whether or not $\boldsymbol{v}_3$ is in C.

Sol. This can be shown by showing that the linear equation $x\boldsymbol{a}_1 + y\boldsymbol{a}_2 = \boldsymbol{v}_3$ is inconsistent, i.e., it does not have a solution. But observe that $A\boldsymbol{v}_1 = -6\boldsymbol{v}_2$, $A\boldsymbol{v}_2 = 9\boldsymbol{v}_2$ and $A\boldsymbol{v}_3 = \mathbf{0}$. So $-6\boldsymbol{v}_1$ and $9\boldsymbol{v}_2$, hence $\boldsymbol{v}_1, \boldsymbol{v}_2$ are in $C = \mathrm{Im}(T)$. Since $\mathrm{rank}(T) = 2$, it is impossible for C to contain $\boldsymbol{v}_3$, as S is a basis of $\boldsymbol{R}^3$. ■

(f) Let $B = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the standard basis of $\boldsymbol{R}^3$. Find $[I]_{S,B}$, where $I : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ ($\boldsymbol{x} \mapsto \boldsymbol{x}$) (the identity operator).

Sol.

$$[I]_{S,B} = [[\boldsymbol{e}_1]_S, [\boldsymbol{e}_2]_S, [\boldsymbol{e}_3]_S] = B^{-1} = \begin{bmatrix} 2/9 & -1/9 & 1/18 \\ 1/27 & 1/27 & 1/27 \\ -20/27 & -2/27 & 5/54 \end{bmatrix}.$$

Note that to find $[\boldsymbol{e}_1]_S$, we need to express $\boldsymbol{e}_1$ as a linear combination of S. Hence we need the solution of $B\boldsymbol{x} = \boldsymbol{e}_1$. So $\boldsymbol{x} = B^{-1}\boldsymbol{e}_1$, which is the first column of B^{-1}. Similarly $[\boldsymbol{e}_2]_S$ is the second column of B^{-1} and $[\boldsymbol{e}_3]_S$ the third. ■

(g) Find $[T]_S$, the matrix for T with respect to the basis S.

Sol. As we have seen above, $A\boldsymbol{v}_1 = -6\boldsymbol{v}_2$, $A\boldsymbol{v}_2 = 9\boldsymbol{v}_2$ and $A\boldsymbol{v}_3 = \boldsymbol{0}$. Hence

$$[T]_S = [[T(\boldsymbol{v}_1)]_S, [T(\boldsymbol{v}_2)]_S, [T(\boldsymbol{v}_3)]_S] = [[-6\boldsymbol{v}_1]_S, [9\boldsymbol{v}_2]_S, [\boldsymbol{0}]_S] = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ ■

3. Recall the definition of an inner product:

> An *inner product* on a real vector space V is a function that associates a real number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ with each pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in V in such a way that the following axioms are satisfied for all vectors $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ in V and all scalars k.
>
> (a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$ (Symmetry axiom)
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> (b) $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{z} \rangle = \langle \boldsymbol{u}, \boldsymbol{z} \rangle + \langle \boldsymbol{v}, \boldsymbol{z} \rangle$ (Additive axiom)
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> (c) $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ (Homogeneity axiom)
>
> (d)

(a) The condition (d) is missing in the definition of an inner product above. State it.

Sol. $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$ and if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$. ■

(b) Give an example of an inner product on a real vector space, which is different from the usual Euclidean Inner Product, and show that it satisfies the conditions above.

Sol. In $\boldsymbol{R}^n$, $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 2\boldsymbol{u}^T \cdot \boldsymbol{v}$ is a inner product. (a)-(d) are easily checked. ■

(c) In an inner product space V, show

$$d(\boldsymbol{u}, \boldsymbol{v}) \leq d(\boldsymbol{u}, \boldsymbol{w}) + d(\boldsymbol{w}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w} \in V.$$

Sol. By definition it is equivalent to the following.

$$\|\boldsymbol{u} - \boldsymbol{v}\| \leq \|\boldsymbol{u} - \boldsymbol{w}\| + \|\boldsymbol{w} - \boldsymbol{v}\|.$$

Set $\boldsymbol{a} = \boldsymbol{u} - \boldsymbol{w}$ and $\boldsymbol{b} = \boldsymbol{w} - \boldsymbol{v}$. Since $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{u} - \boldsymbol{v}$, it suffices to show that $\|\boldsymbol{a} + \boldsymbol{b}\| \leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$, or $\|\boldsymbol{a} + \boldsymbol{b}\|^2 \leq (\|\boldsymbol{a}\| + \|\boldsymbol{b}\|)^2$. Now

$$\begin{aligned}
&(\|\boldsymbol{a}\| + \|\boldsymbol{b}\|)^2 - \|\boldsymbol{a} + \boldsymbol{b}\|^2 \\
&= \|\boldsymbol{a}\|^2 + 2\|\boldsymbol{a}\|\|\boldsymbol{b}\| + \|\boldsymbol{b}\|^2 - \langle \boldsymbol{a} + \boldsymbol{b}, \boldsymbol{a} + \boldsymbol{b} \rangle \\
&= \|\boldsymbol{a}\|^2 + 2\|\boldsymbol{a}\|\|\boldsymbol{b}\| + \|\boldsymbol{b}\|^2 - (\|\boldsymbol{a}\|^2 + 2\langle \boldsymbol{a}, \boldsymbol{b} \rangle + \|\boldsymbol{b}\|^2) \\
&= 2(\|\boldsymbol{a}\|\|\boldsymbol{b}\| - \langle \boldsymbol{a}, \boldsymbol{b} \rangle) \geq 0
\end{aligned}$$

by Theorem 2. Hence we have shown the inequality. ■

4. Let V be an n-dimensional vector space, W_1 and W_2 subspaces of V. Set $U = \{\boldsymbol{w}_1 + \boldsymbol{w}_2 \mid \boldsymbol{w}_1 \in W_1 \text{ and } \boldsymbol{w}_2 \in W_2\}$. ($U$ is often denoted by $W_1 + W_2$.)

   (a) Show that $W = W_1 \cap W_2 = \{\boldsymbol{w} \mid \boldsymbol{w} \in W_1 \text{ and } \boldsymbol{w} \in W_2\}$ is a subspace of V.

   Sol. Let $\boldsymbol{u}, \boldsymbol{u}' \in W_1 \cap W_2$. Then $\boldsymbol{u}, \boldsymbol{u}' \in W_1$ and $\boldsymbol{u}, \boldsymbol{u}' \in W_2$. Since W_1 and W_2 are subspaces, $\boldsymbol{u} + \boldsymbol{u}' \in W_1$, $\boldsymbol{u} + \boldsymbol{u}' \in W_2$. Hence $\boldsymbol{u} + \boldsymbol{u}' \in W_1 \cap W_2$. Similarly $k\boldsymbol{u} \in W$ and $k\boldsymbol{u} \in W_2$ for all scalars k. Hence $k\boldsymbol{u} \in W_1 \cap W_2$. Therefore, $W_1 \cap W_2$ is a subspace. (See Theorem 3.2.) ■

   (b) Show that U is a subspace of V.

   Sol. Let $\boldsymbol{u}, \boldsymbol{u}' \in U$. Then there exist $\boldsymbol{w}_1, \boldsymbol{w}_1' \in W_1$ and $\boldsymbol{w}_2, \boldsymbol{w}_2' \in W_2$ such that $\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2$ and $\boldsymbol{u}' = \boldsymbol{w}_1' + \boldsymbol{w}_2'$. Since $\boldsymbol{w}_1, \boldsymbol{w}_1' \in W_1$ and $\boldsymbol{w}_2, \boldsymbol{w}_2' \in W_2$ and W_1 and W_2 are subspaces, $\boldsymbol{w}_1 + \boldsymbol{w}_1' \in W_1$, $\boldsymbol{w}_2 + \boldsymbol{w}_2' \in W_2$. Hence $\boldsymbol{u} + \boldsymbol{u}' = (\boldsymbol{w}_1 + \boldsymbol{w}_1') + (\boldsymbol{w}_2 + \boldsymbol{w}_2') \in U$. Similarly $k\boldsymbol{u} = k\boldsymbol{w}_1 + k\boldsymbol{w}_2 \in U$ as $k\boldsymbol{w}_1 \in W_1$ and $k\boldsymbol{w}_2 \in W_2$. ■

   (c) Show that there is a set of vectors

   $$S = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s, \boldsymbol{w}_{s+1}, \boldsymbol{w}_{s+2}, \ldots, \boldsymbol{w}_{s+t}, \boldsymbol{w}_{s+t+1}, \boldsymbol{w}_{s+t+2}, \ldots, \boldsymbol{w}_{s+t+r}\}$$

   of V satisfying the following conditions.

   i. $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s\}$ is a basis of $W = W_1 \cap W_2$,

   ii. $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s, \boldsymbol{w}_{s+1}, \boldsymbol{w}_{s+2}, \ldots, \boldsymbol{w}_{s+t}\}$ is a basis of W_1; and

   iii. $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s, \boldsymbol{w}_{s+t+1}, \boldsymbol{w}_{s+t+2}, \ldots, \boldsymbol{w}_{s+t+r}\}$ is a basis of W_2.

   Sol. Since W, W_1, W_2 are all subspaces of V, these spaces are finite dimensional by Theorem 1 (d), and these have bases. First take a basis $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s\}$ of W. (i). Since $W \subset W_1$, and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s\}$ is linearly independent, it can be enlarged to a basis of W_1 by inserting appropriate vectors $\boldsymbol{w}_{s+1}, \boldsymbol{w}_{s+2}, \ldots, \boldsymbol{w}_{s+t}$ into $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s\}$. We applied Theorem 1 (c). Hence (ii). The condition (iii) is similar by inserting $\boldsymbol{w}_{s+t+1}, \boldsymbol{w}_{s+t+2}, \ldots, \boldsymbol{w}_{s+t+r}$. ■

   (d) Show that $U = \mathrm{Span}(S)$.

   Sol. Every vector of U is a sum of a vector $\boldsymbol{u}_1$ in W_1 and a vector $\boldsymbol{u}_2$ in W_2. Since $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s, \boldsymbol{w}_{s+1}, \boldsymbol{w}_{s+2}, \ldots, \boldsymbol{w}_{s+t}\} \subset S$ is a basis of W_1, $\boldsymbol{u}_1$ is a linear combination of these vectors and hence it is in $\mathrm{Span}(S)$. Similarly $\boldsymbol{u}_2 \in \mathrm{Span}(S)$. Since $\mathrm{Span}(S)$ is a subspace (Theorem 3.4), $\boldsymbol{u}_1 + \boldsymbol{u}_2 \in \mathrm{Span}(S)$ and $U \subset \mathrm{Span}(S)$. Since $S \subset U$, and U is a subspace, $\mathrm{Span}(S) \subset U$. We have $U = \mathrm{Span}(S)$. ■

(e) Show that S is a basis of U.

Sol. It suffices to show that S is linearly independent. Suppose

$$\begin{aligned}\mathbf{0} &= a_1\boldsymbol{w}_1 + a_2\boldsymbol{w}_2 + \cdots + a_s\boldsymbol{w}_s + a_{s+1}\boldsymbol{w}_{s+1} + a_{s+2}\boldsymbol{w}_{s+2} \\ &\quad + \cdots + a_{s+t}\boldsymbol{w}_{s+t} + a_{s+t+1}\boldsymbol{w}_{s+t+1} + a_{s+t+2}\boldsymbol{w}_{s+t+2} + \cdots + a_{s+t+r}\boldsymbol{w}_{s+t+r}.\end{aligned}$$

Consider

$$\begin{aligned}& a_1\boldsymbol{w}_1 + a_2\boldsymbol{w}_2 + \cdots + a_s\boldsymbol{w}_s + a_{s+1}\boldsymbol{w}_{s+1} + a_{s+2}\boldsymbol{w}_{s+2} + \cdots + a_{s+t}\boldsymbol{w}_{s+t} \\ = \ & -(a_{s+t+1}\boldsymbol{w}_{s+t+1} + a_{s+t+2}\boldsymbol{w}_{s+t+2} + \cdots + a_{s+t+r}\boldsymbol{w}_{s+t+r}).\end{aligned}$$

Since the left hand side is in W_1 and the right hand side is in W_2, it is in $W = W_1 \cap W_2$. So it can be written as a linear combination of $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s\}$. Set

$$\begin{aligned}& a_1\boldsymbol{w}_1 + a_2\boldsymbol{w}_2 + \cdots + a_s\boldsymbol{w}_s + a_{s+1}\boldsymbol{w}_{s+1} + a_{s+2}\boldsymbol{w}_{s+2} + \cdots + a_{s+t}\boldsymbol{w}_{s+t} \\ = \ & -(a_{s+t+1}\boldsymbol{w}_{s+t+1} + a_{s+t+2}\boldsymbol{w}_{s+t+2} + \cdots + a_{s+t+r}\boldsymbol{w}_{s+t+r}) \\ = \ & c_1\boldsymbol{w}_1 + c_2\boldsymbol{w}_2 + \cdots + c_s\boldsymbol{w}_s.\end{aligned}$$

By the uniqueness of expression in W_1 proved in Problem 1, by equating the first line with the third line, we have $a_{s+1} = a_{s+2} = \cdots = a_{s+t} = 0$. Now we have

$$a_1\boldsymbol{w}_1 + a_2\boldsymbol{w}_2 + \cdots + a_s\boldsymbol{w}_s + a_{s+1}\boldsymbol{w}_{s+1} + a_{s+2}\boldsymbol{w}_{s+2} + \cdots + a_{s+t}\boldsymbol{w}_{s+t} = \mathbf{0}$$

and $a_1 = a_2 = \cdots = a_s = a_{s+1} = a_{s+2} = \cdots = a_{s+t} = 0$ as $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_s, \boldsymbol{w}_{s+1}, \boldsymbol{w}_{s+2}, \ldots, \boldsymbol{w}_{s+t}\}$ is a basis of W_1 and is linearly independent. ■

Problem 4 proves that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.