Take-Home **Quiz 1** (*Due at 7:00 p.m. on Wed. Dec. 13, 2006*)

Division: **ID#:** **Name:**

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be non-zero vectors in $\boldsymbol{R}^n$.

1. Let λ be a real number. Show the following. (Hint: use $\|\boldsymbol{w}\|^2 = \boldsymbol{w} \cdot \boldsymbol{w}$.)

$$\|\lambda \boldsymbol{u} + \boldsymbol{v}\|^2 = \lambda^2 \|\boldsymbol{u}\|^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v})\lambda + \|\boldsymbol{v}\|^2.$$

2. Using the fact that $\|\lambda \boldsymbol{u} + \boldsymbol{v}\|^2 \geq 0$ for all real λ and a property of a quadratic function, show the Cauchy-Schwarz Inequality. (Hint: Discriminant (*Hanbetsu-shiki*))

3. Show the equivalence of the following:

$$|\boldsymbol{u} \cdot \boldsymbol{v}| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \Leftrightarrow \text{There exists } \alpha \in \boldsymbol{R} \text{ such that } \boldsymbol{u} = \alpha \boldsymbol{v}.$$

Message: (1) この授業を履修した理由 (2) この授業に期待すること [HP 掲載不可のときは明記のこと]

Solutions to Take-Home Quiz 1 (*December 13, 2006*)

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be non-zero vectors in $\boldsymbol{R}^n$.

1. Let λ be a real number. Show the following. (Hint: use $\|\boldsymbol{w}\|^2 = \boldsymbol{w} \cdot \boldsymbol{w}$.)

$$\|\lambda\boldsymbol{u} + \boldsymbol{v}\|^2 = \lambda^2\|\boldsymbol{u}\|^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v})\lambda + \|\boldsymbol{v}\|^2.$$

Sol.

$$\begin{aligned}\|\lambda\boldsymbol{u} + \boldsymbol{v}\|^2 &= (\lambda\boldsymbol{u} + \boldsymbol{v}) \cdot (\lambda\boldsymbol{u} + \boldsymbol{v}) \\ &= (\boldsymbol{u} \cdot \boldsymbol{u})\lambda^2 + (\boldsymbol{u} \cdot \boldsymbol{v})\lambda + (\boldsymbol{v} \cdot \boldsymbol{u})\lambda + (\boldsymbol{v} \cdot \boldsymbol{v}) \\ &= \|\boldsymbol{u}\|^2\lambda^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v})\lambda + \|\boldsymbol{v}\|^2.\end{aligned}$$

2. Using the fact that $\|\lambda\boldsymbol{u}+\boldsymbol{v}\|^2 \geq 0$ for all real λ and a property of a quadratic function, show the Cauchy-Schwarz Inequality. (Hint: Discriminant (*Hanbetsu-shiki*))

Sol. Note that $\|\boldsymbol{u}\| \neq 0$ implies that the right hand side above is a polynomial of degree 2. Since the right hand side of the equation in 1 is quadratic in λ, it can be considered as a quadratic function which takes only nonnegative values for all real λ. Hence the graph of the function is above the x-axis or possibly the vertex of the parabola touches the x-axis. Hence the equation $\|\lambda\boldsymbol{u} + \boldsymbol{v}\|^2 = 0$ has either no real solutions or exactly one solution. Therefore the discriminant of it is nonpositive and we have

$$(\boldsymbol{u} \cdot \boldsymbol{v})^2 - \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 \leq 0.$$

Thus we have $(\boldsymbol{u} \cdot \boldsymbol{v})^2 \leq \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2$, or

$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|.$$

This is the Cauchy-Sshwarz Inequality.

Although we assumed that both $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors, the Cauchy-Schwarz Inequality holds even if one of them is a zero vector. So it is easy to check that the equality holds for all cases.

3. Show the equivalence of the following:

$$|\boldsymbol{u} \cdot \boldsymbol{v}| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \Leftrightarrow \text{There exists } \alpha \in \boldsymbol{R} \text{ such that } \boldsymbol{u} = \alpha\boldsymbol{v}.$$

Sol. If the equality holds, the discriminant is zero. Hence the vertex of the parabola touches the x-axis. That means there is a value λ such that $\|\lambda\boldsymbol{u}+\boldsymbol{v}\| = 0$. Hence $\lambda\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{0}$. If $\lambda = 0$, then $\boldsymbol{v} = \boldsymbol{0}$, a contradiction. Hence $\lambda \neq 0$. Let $\alpha = -(1/\lambda)$. Then $\boldsymbol{u} = \alpha\boldsymbol{v}$ as desired.

Take-Home **Quiz 2** (*Due at 7:00 p.m. on Wed. Dec. 20, 2006*)

Division: **ID#:** **Name:**

For $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^T$ be a nonzero vector in $\boldsymbol{R}^n$, Let

$$\tau_{\boldsymbol{u}} : \boldsymbol{R}^n \to \boldsymbol{R}^n \ (\boldsymbol{x} \mapsto \boldsymbol{x} - \frac{2\boldsymbol{x} \cdot \boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}).$$

1. Show that $\tau_{\boldsymbol{u}}$ is a linear transformation.

2. Let $\boldsymbol{v} = (1, -1, 0, \ldots, 0)^T$. Find the standard matrix $[\tau_{\boldsymbol{v}}]$.

3. Suppose T is a linear transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$ such that $T(\boldsymbol{u}) = -\boldsymbol{u}$, $T(\boldsymbol{w}) = \boldsymbol{w}$ whenever $\boldsymbol{w} \cdot \boldsymbol{u} = 0$. Show that $T = \tau_{\boldsymbol{u}}$. (Hint: If $\alpha = \frac{\boldsymbol{x} \cdot \boldsymbol{u}}{\|\boldsymbol{u}\|^2}$, $(\boldsymbol{x} - \alpha\boldsymbol{u}) \cdot \boldsymbol{u} = 0$.)

Message 欄：(人それぞれの関わり方がある中で）高校・大学における数学は何のため？
[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 2 (*December 20, 2006*)

For $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^T$ be a nonzero vector in $\boldsymbol{R}^n$, Let

$$\tau_{\boldsymbol{u}} : \boldsymbol{R}^n \to \boldsymbol{R}^n \ (\boldsymbol{x} \mapsto \boldsymbol{x} - \frac{2\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}).$$

1. Show that $\tau_{\boldsymbol{u}}$ is a linear transformation. (This linear transformation is called the *reflection* defined by $\boldsymbol{u}$.)

   Sol. Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^n$ and k a scalar. Then

$$\tau_{\boldsymbol{u}}(\boldsymbol{x}+\boldsymbol{y}) = (\boldsymbol{x}+\boldsymbol{y}) - \frac{2(\boldsymbol{x}+\boldsymbol{y})\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u} = \Big(\boldsymbol{x} - \frac{2\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}\Big) + \Big(\boldsymbol{y} - \frac{2\boldsymbol{y}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}\Big) = \tau_{\boldsymbol{u}}(\boldsymbol{x}) + \tau_{\boldsymbol{u}}(\boldsymbol{y})$$

$$\tau_{\boldsymbol{u}}(k\boldsymbol{x}) = k\boldsymbol{x} - \frac{2(k\boldsymbol{x})\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u} = k\Big(\boldsymbol{x} - \frac{2\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}\Big) = k\tau_{\boldsymbol{u}}(\boldsymbol{x}).$$

   Hence $\tau_{\boldsymbol{u}}$ is a linear transformation by Theorem 4.3.2 in the textbook. ■

2. Let $\boldsymbol{v} = (1, -1, 0, \ldots, 0)^T$. Find the standard matrix $[\tau_{\boldsymbol{v}}]$.

   Sol. Let $\boldsymbol{e}_1 = (1, 0, \ldots, 0)^T$, $\boldsymbol{e}_2 = (0, 1, 0, \ldots, 0)^T, \ldots, \boldsymbol{e}_n = (0, \ldots, 0, 1)^T$ be unit vectors. Since $\|\boldsymbol{v}\|^2 = 2$, $\tau_{\boldsymbol{v}}(\boldsymbol{e}_1) = (0, 1, 0, \ldots, 0)^T$, $\tau_{\boldsymbol{v}}(\boldsymbol{e}_2) = (1, 0, \ldots, 0)^T$, and $\tau_{\boldsymbol{v}}(\boldsymbol{e}_i) = \boldsymbol{e}_i$ if $i = 3, 4, \ldots, n$. We have

$$[\tau_{\boldsymbol{v}}] = [\tau_{\boldsymbol{v}}(\boldsymbol{e}_1), \tau_{\boldsymbol{v}}(\boldsymbol{e}_2), \tau_{\boldsymbol{v}}(\boldsymbol{e}_3), \ldots, \tau_{\boldsymbol{v}}(\boldsymbol{e}_n)] = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

   by Theorem 4.3.3. ■

3. Suppose T is a linear transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$ such that $T(\boldsymbol{u}) = -\boldsymbol{u}$, $T(\boldsymbol{w}) = \boldsymbol{w}$ whenever $\boldsymbol{w}\cdot\boldsymbol{u} = 0$. Show that $T = \tau_{\boldsymbol{u}}$. (Hint: If $\alpha = \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}$, $(\boldsymbol{x} - \alpha\boldsymbol{u})\cdot\boldsymbol{u} = 0$.)

   Sol. Since both T and $\tau_{\boldsymbol{u}}$ are linear transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$. It remains to show that $T(\boldsymbol{x}) = \tau_{\boldsymbol{u}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{R}^n$. Since

$$(*) \quad (\boldsymbol{x} - \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u})\cdot\boldsymbol{u} = 0 \ \text{ and } \ (**) \quad \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2} \text{ is a scalar,}$$

$$\begin{aligned} T(\boldsymbol{x}) &= T((\boldsymbol{x} - \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}) + \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}) \\ &= T(\boldsymbol{x} - \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}) + \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}T(\boldsymbol{u}) \quad \text{(by Theorem 4.3.2 (a) and (b) with (**))} \\ &= \Big(\boldsymbol{x} - \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}\Big) - \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u} \quad \text{(by the properties of } T \text{ and (*) above)} \\ &= \boldsymbol{x} - \frac{2\boldsymbol{x}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u} \\ &= \tau_{\boldsymbol{u}}(\boldsymbol{x}). \end{aligned}$$

   Therefore $T = \tau_{\boldsymbol{u}}$ as functions (or mappings). ■

Take-Home **Quiz 3** (*Due at 7:00 p.m. on Wed. January 10, 2007*)

Division: **ID#:** **Name:**

1. Let V be a vector space and k a scalar. Show $k\mathbf{0} = \mathbf{0}$. In each step of your proof quote the axiom applied. [Hint: Exercise 5.1.29]

2. Let A, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be as follows.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix},\ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -8 \\ -6 \end{bmatrix}.$$

(a) Let $B = (A - I)^2$. Show that $W = \{\boldsymbol{v} \in \boldsymbol{R}^3 \mid B\boldsymbol{v} = 10\boldsymbol{v}\}$ is a subspace of $V = \boldsymbol{R}^3$.

(b) Determine whether or not $\boldsymbol{v}_3$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Message 欄：今年の抱負、将来の夢。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 3 (*January 10, 2007*)

1. Let V be a vector space and k a scalar. Show $k\mathbf{0} = \mathbf{0}$. In each step of your proof quote the axiom applied. [Hint: Exercise 5.1.29]

   Sol.

$$\begin{aligned} k\mathbf{0} &\overset{(4)}{=} k\mathbf{0} + \mathbf{0} \overset{(5)}{=} k\mathbf{0} + (k\mathbf{0} + (-(k\mathbf{0})) \overset{(3)}{=} (k\mathbf{0} + k\mathbf{0}) + (-(k\mathbf{0})) \\ &\overset{(7)}{=} k(\mathbf{0} + \mathbf{0}) + (-(k\mathbf{0})) \overset{(4)}{=} k\mathbf{0} + (-(k\mathbf{0})) \overset{(5)}{=} \mathbf{0}. \end{aligned}$$

   Therefore $k\mathbf{0} = \mathbf{0}$ ■

   We write $\boldsymbol{u} - \boldsymbol{v}$ for $\boldsymbol{u} + (-\boldsymbol{v})$. Note that since $k\mathbf{0}$ is an element in a vector space V, $-(k\mathbf{0})$ above is an element guaranteed to exist by Axiom 5.

2. Let A, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be as follows.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix},\ I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -8 \\ -6 \end{bmatrix}.$$

   (a) Let $B = (A - I)^2$. Show that $W = \{\boldsymbol{v} \in \boldsymbol{R}^3 \mid B\boldsymbol{v} = 10\boldsymbol{v}\}$ is a subspace of $V = \boldsymbol{R}^3$.

   Sol. Since $W = \{\boldsymbol{v} \in \boldsymbol{R}^3 \mid (B - 10I)\boldsymbol{v} = \mathbf{0}\}$, W is the kernel of the linear transformation defined by a 3×3 matrix $B - 10I$. Hence W is a subspace of V by Proposition 3.3 (5.2.2). ■

   Alternatively apply Theorem 3.2 (5.2.1). Since $\mathbf{0}$ safisfies $B\mathbf{0} = \mathbf{0} = 10\mathbf{0}$, $\mathbf{0} \in W$. Hence W is not empty. Let $\boldsymbol{u}, \boldsymbol{v} \in W$, i.e., $B\boldsymbol{u} = 10\boldsymbol{u}$ and $B\boldsymbol{v} = 10\boldsymbol{v}$. Let $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$. Then

$$B\boldsymbol{w} = B(\boldsymbol{u} + \boldsymbol{v}) = B\boldsymbol{u} + B\boldsymbol{v} = 10\boldsymbol{u} + 10\boldsymbol{v} = 10(\boldsymbol{u} + \boldsymbol{v}) = 10\boldsymbol{w}.$$

   Hence $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{w} \in W$. Similarly if k is a scalar

$$B(k\boldsymbol{u}) = k(B\boldsymbol{u}) = k(10\boldsymbol{u}) = 10(k\boldsymbol{u}).$$

   Hence $k\boldsymbol{u} \in W$. Thus W is a subspace of V by Theorem 3.2 (5.2.1) and W itself is a vector space. ■

   (b) Determine whether or not $\boldsymbol{v}_3$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

   Sol. Since $\boldsymbol{v}_3 = 5\boldsymbol{v}_1 + \boldsymbol{v}_2$, $\boldsymbol{v}_3$ can be written as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. ■

   Let $\boldsymbol{v}_3 = x\boldsymbol{v}_1 + y\boldsymbol{v}_2$. Then the augmented (or extended coefficient) matrix of this system of linear equations is A. Hence by applying elementary row operations we have

$$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Now we have the linear combination above.

Take-Home # Quiz 4 *(Due at 7:00 p.m. on Wed. January 17, 2007)*

Division: **ID#:** **Name:**

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ be vectors in $\boldsymbol{R}^3$ given below.

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -8 \\ -6 \end{bmatrix},\ \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

1. Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis of $U = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

2. Show that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a basis of $\boldsymbol{R}^3$.

3. Show that $\boldsymbol{e}_1 \notin \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

4. Show that $\{\boldsymbol{e}_1, \boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis of $\boldsymbol{R}^3$.

5. Express $\boldsymbol{e}_2$ as a linear combination of $\boldsymbol{e}_1, \boldsymbol{v}_1, \boldsymbol{v}_2$.

Message 欄：あなたにとって、豊かな生活とはどのようなものでしょうか。どのようなとき幸せだと感じますか。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 4 (*January 17, 2007*)

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ be vectors in $\boldsymbol{R}^3$ given below.

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -8 \\ -6 \end{bmatrix}, \ \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

N.B. A subspace of a vector space is a vector space. A subset $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_r\}$ of a vector space V is a basis of V, whenever two conditions are satisfied, i.e., '(a) linear independence' and '(b) $V = \mathrm{Span}(S)$'. Review the definition of a basis.

1. Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis of $U = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

   Sol. Note that $\boldsymbol{v}_3 = 5\boldsymbol{v}_1 + \boldsymbol{v}_2$. (See Quiz 3.) Hence

   $$U = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} = \{a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 \mid a_1, a_2, a_3 \in \boldsymbol{R}\} = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}.$$

   By definition, it suffices to show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a linearly independent set. Suppose

   $$\mathbf{0} = x\boldsymbol{v}_1 + y\boldsymbol{v}_2 = x \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + y \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} x - 2y \\ -3x + 7y \\ -2x + 4y \end{bmatrix}.$$

   Since $x - 2y = 0$ and $-3x + 7y = 0$ implies $x = y = 0$, $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a linearly independent set and it is a basis of U. ∎

2. Show that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a basis of $\boldsymbol{R}^3$.

   Sol. Suppose $\mathbf{0} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3 = (x, y, z)^T$. Then $x = y = z = 0$. Hence $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is linearly independent. Moreover $(x, y, z)^T = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$ for all $x, y, z \in \boldsymbol{R}$, and $\boldsymbol{R}^3 = \mathrm{Span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Hence $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a basis of $\boldsymbol{R}^3$. ∎

3. Show that $\boldsymbol{e}_1 \notin \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

   Sol. By the computation in 1, the third entry of $x\boldsymbol{v}_1 + y\boldsymbol{v}_2$ is -2 times the first entry. Hence $\boldsymbol{e}_1$ is not in $U = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. ∎

4. Show that $\{\boldsymbol{e}_1, \boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis of $\boldsymbol{R}^3$.

   Sol. By 3, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{e}_1\}$ is a linearly independent set in $\boldsymbol{R}^3$. (See Proposition 4.7 (5.4.4).) Since $\dim(\boldsymbol{R}^3) = 3$ by 2. $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{e}_1\}$ is a basis of $\boldsymbol{R}^3$ by Theorem 4.8 (5.4.5). ∎

5. Express $\boldsymbol{e}_2$ as a linear combination of $\boldsymbol{e}_1, \boldsymbol{v}_1, \boldsymbol{v}_2$.

   Sol. $\boldsymbol{e}_2 = 0\boldsymbol{e}_1 + 2\boldsymbol{v}_1 + \boldsymbol{v}_2 = 2\boldsymbol{v}_1 + \boldsymbol{v}_2$.

∎

Take-Home **Quiz 5** (*Due at 7:00 p.m. on Wed. January 24, 2007*)

Division: **ID#:** **Name:**

Let A be the coefficient matrix, and B the augmented matrix of a system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{x} = [x_1, x_2, x_3, x_4, x_5, x_6]^T$. Let C be a reduced row-echelon form obtained from B by a series of elementary row operations.

$$B = [A, \boldsymbol{b}] = \begin{bmatrix} 3 & -3 & 0 & 3 & -6 & -3 & -6 \\ 2 & -2 & 1 & 5 & -3 & -1 & 1 \\ -3 & 3 & 0 & -3 & 6 & 1 & -8 \\ -1 & 1 & 2 & 5 & 4 & 0 & -9 \end{bmatrix} \to C = \begin{bmatrix} 1 & -1 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 3 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank(A) and nullity(A).

2. Find a basis of the row space of A.

3. Find a basis of the column space of A.

4. Find a basis of the nullspace of A.

5. Find the general solution of the equation $A\boldsymbol{x} = \boldsymbol{b}$.

Message 欄：これまでの Linear Algebra II について。改善点について。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 5 (*January 24, 2007*)

Let A be the coefficient matrix, and B the augmented matrix of a system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{x} = [x_1, x_2, x_3, x_4, x_5, x_6]^T$. Let C be a reduced row-echelon form obtained from B by a series of elementary row operations.

$$B = [A, \boldsymbol{b}] = \begin{bmatrix} 3 & -3 & 0 & 3 & -6 & -3 & -6 \\ 2 & -2 & 1 & 5 & -3 & -1 & 1 \\ -3 & 3 & 0 & -3 & 6 & 1 & -8 \\ -1 & 1 & 2 & 5 & 4 & 0 & -9 \end{bmatrix} \to C = \begin{bmatrix} 1 & -1 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 3 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the following, let $C = [D, \boldsymbol{e}]$, where $\boldsymbol{e} = [5, -2, 7, 0]^T$. Then there is an invertible matrix P of size 4×4 such that $PB = C$ and $PA = D$, $P\boldsymbol{b} = \boldsymbol{e}$. For a matrix M, let M_i denote column i of M.

1. Find rank(A) and nullity(A).

   Sol. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ be the standard basis of $\boldsymbol{R}^4$. Then $D_1 = \boldsymbol{e}_1, D_3 = \boldsymbol{e}_2$ and $D_6 = \boldsymbol{e}_3$. Hence

   $$\text{rank}(A) = \dim(\mathcal{C}(A)) = \dim(\mathcal{C}(PA)) = \dim(\mathcal{C}(D)) = \dim \text{Span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} = 3.$$

   Since $\mathcal{N}(A) = \mathcal{N}(PA) = \mathcal{N}(D)$, nullity($A$) = nullity($PA$) = nullity($D$) = 3. See Problem 4, or use Theorem 5.6 (5.6.3). ■

2. Find a basis of the row space of A.

   Sol. Since $\mathcal{R}(A) = \mathcal{R}(D)$ by Proposition 5.2 (5.5.4), it suffices to find a basis of the row space of D. Let $S = \{[1, -1, 0, 1, -2, 0], [0, 0, 1, 3, 1, 0], [0, 0, 0, 0, 1]\}$. Then clearly $\text{Span}(S) = \mathcal{R}(D)$, and S is a linearly independent set. Hence S is a basis. ■

3. Find a basis of the column space of A.

   Sol. Since $PA_1 = D_1 = \boldsymbol{e}_1$, $PA_3 = D_3 = \boldsymbol{e}_2$ and $PA_6 = D_6 = \boldsymbol{e}_3$ form a basis of $\mathcal{C}(D)$, $\{A_1, A_3, A_6\}$ is a linearly independent set by Proposition 5.3 (5.5.5). Since $\text{rank}(A) = \dim(\mathcal{C}(A)) = 3$, $\{A_1, A_3, A_6\}$ is a basis of the column space of A. ■

4. Find a basis of the nullspace of A.

   Sol. Since $\mathcal{N}(A) = \mathcal{N}(PA) = \mathcal{N}(D)$ and $\{[1, 1, 0, 0, 0, 0], [-1, 0, -3, 1, 0, 0], [2, 0, -1, 0, 1, 0]\}$ is a linearly independent set, this is a basis. ■

5. Find the general solution of the equation $A\boldsymbol{x} = \boldsymbol{b}$.

   Sol.

   $$\begin{bmatrix} 5 \\ 0 \\ -2 \\ 0 \\ 0 \\ 7 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \quad (s,\ t,\ u \text{ are parameters.})$$

   ■

Take-Home **Quiz 6** (*Due at 7:00 p.m. on Wed. January 31, 2007*)

Division: **ID#:** **Name:**

Let A be an $m \times n$ matrix. For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^n$ let

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = A\boldsymbol{u} \cdot A\boldsymbol{v} = (A\boldsymbol{u})^T A\boldsymbol{v} = \boldsymbol{u}^T A^T A\boldsymbol{v}.$$

1. Show that $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ satisfies the properties (a), (b) and (c) of an inner product in Definition 6.1 (or 1, 2, 3 in the definition on page 296 in the textbook).

2. Show that if $\mathcal{N}(A) = \{\boldsymbol{v} \in \boldsymbol{R}^n \mid A\boldsymbol{v} = \boldsymbol{0}\} = \{\boldsymbol{0}\}$, then $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ is an inner product.

3. Show that if $m < n$, then $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ is not an inner product.

4. Show that $m \geq n$, if $A^T A$ is invertible.

Message 欄：数学（または他の科目）など何かを学んでいて感激したことについて。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 6 (*January 31, 2007*)

Let A be an $m \times n$ matrix. For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^n$ let

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = A\boldsymbol{u} \cdot A\boldsymbol{v} = (A\boldsymbol{u})^T A\boldsymbol{v} = \boldsymbol{u}^T A^T A\boldsymbol{v}.$$

1. Show that $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ satisfies the properties (a), (b) and (c) of an inner product in Definition 6.1 (or 1, 2, 3 in the definition on page 296 in the textbook).

   Sol. First note that (a), (b), and (c) hold for $\boldsymbol{u} \cdot \boldsymbol{v}$ in $\boldsymbol{R}^m$. Note that if $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^n$, then $A\boldsymbol{u}$, $A\boldsymbol{v} \in \boldsymbol{R}^m$. See Theorem 1.2 (4.1.2). For if $\boldsymbol{x} = [x_1, x_2, \ldots, x_m]^T$ and $\boldsymbol{y} = [y_1, y_2, \ldots, y_m]^T$, (a) $\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + x_2 y_2 + \cdots + x_m y_m = \boldsymbol{y} \cdot \boldsymbol{x}$, (b) $(\boldsymbol{x} + \boldsymbol{y}) \cdot \boldsymbol{z} = \boldsymbol{x} \cdot \boldsymbol{z} + \boldsymbol{y} \cdot \boldsymbol{z}$, (c) $(k\boldsymbol{x}) \cdot \boldsymbol{y} = k(\boldsymbol{x} \cdot \boldsymbol{y})$. Moreover $\boldsymbol{x} \cdot \boldsymbol{x} \geq 0$ and $\boldsymbol{x} \cdot \boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$.

   (a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = (A\boldsymbol{u}) \cdot (A\boldsymbol{v}) = (A\boldsymbol{v}) \cdot (A\boldsymbol{u}) = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$.

   (b) $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{z} \rangle = (A(\boldsymbol{u} + \boldsymbol{v})) \cdot (A\boldsymbol{z}) = (A\boldsymbol{u}) \cdot (A\boldsymbol{z}) + (A\boldsymbol{v}) \cdot (A\boldsymbol{z}) = \langle \boldsymbol{u}, \boldsymbol{z} \rangle + \langle \boldsymbol{v}, \boldsymbol{z} \rangle$.

   (c) $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = (Ak\boldsymbol{u}) \cdot (A\boldsymbol{v}) = k((A\boldsymbol{u}) \cdot (A\boldsymbol{v})) = k\langle \boldsymbol{u}, \boldsymbol{v} \rangle$. ■

2. Show that if $\mathcal{N}(A) = \{\boldsymbol{v} \in \boldsymbol{R}^n \mid A\boldsymbol{v} = \boldsymbol{0}\} = \{\boldsymbol{0}\}$, then $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ is an inner product.

   Sol. It suffices to show the condition (d). Clearly, $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = (A\boldsymbol{u}) \cdot (A\boldsymbol{u}) \geq 0$. If $A\boldsymbol{u} = [w_1, w_2, \ldots, w_n]^T$, then $(A\boldsymbol{u}) \cdot (A\boldsymbol{u}) = 0$ if and only if $A\boldsymbol{u} = \boldsymbol{0}$ if and only if $\boldsymbol{u} \in \mathcal{N}(A)$. Hence if the condition above is satisfied, then $\boldsymbol{u} = \boldsymbol{0}$. Thus $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = (A\boldsymbol{u}) \cdot (A\boldsymbol{u}) = 0$ implies $\boldsymbol{u} = \boldsymbol{0}$ and $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ safisfies all conditions of an inner product in Definition 6.1. ■

3. Show that if $m < n$, then $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ is not an inner product.

   Sol. If $m < n$, then by Theorem 4.3 (5.3.3), the system of linear equation $A\boldsymbol{x} = \boldsymbol{0}$ has a nonzero solution $\boldsymbol{u}$. Then $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = A\boldsymbol{u} \cdot A\boldsymbol{u} = 0$ while $\boldsymbol{u} \neq \boldsymbol{0}$. Thus $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ does not satisfy (d) and it is not an inner product. ■

4. Show that $m \geq n$, if $A^T A$ is invertible.

   Sol. Suppose $m < n$. Then there exists a nonzero vector $\boldsymbol{u} \in \boldsymbol{R}^n$ such that $A\boldsymbol{u} = \boldsymbol{0}$ by Theorem 4.3 (5.3.3). Then $A^T A\boldsymbol{u} = A^T\boldsymbol{0} = \boldsymbol{0}$. Since $A^T A$ is invertible, $\boldsymbol{u} = \boldsymbol{0}$, a contradiction. Hence $m \geq n$. ■

N.B. Two kinds of zero $\boldsymbol{0}$ and 0 are used above. But actually there are three. Some of $\boldsymbol{0}$ are $\boldsymbol{0}_n \in \boldsymbol{R}^n$ and the others are $\boldsymbol{0}_m \in \boldsymbol{R}^m$. Can you identify them?

Take-Home **Quiz 7** (*Due at 7:00 p.m. on Wed. February 7, 2007*)

Division: **ID#:** **Name:**

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ be vectors in $\boldsymbol{R}^3$ given below.

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -8 \\ -6 \end{bmatrix}, \ \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^3$, let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v}$ be the inner product and $U = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. You may quote the facts shown in previous quizzes.

1. Compute $\boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2} \boldsymbol{v}_1$.

2. Find an orthonormal basis of U.

3. Find an orthonormal basis of $\boldsymbol{R}^3$ containing the basis constructed in 2.

4. Find a basis of $U^\perp$.

5. Express each of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ as a linear combination of the orthonormal basis constructed in 3.

Message 欄：ICU をどのようにして知りましたか。選んだ理由。ICU の入試について。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 7 (*February 6, 2007*)

1. Compute $\boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$.

   Sol. $\langle \boldsymbol{v}_2, \boldsymbol{v}_1\rangle = -2 - 21 - 8 = -31$, $\|\boldsymbol{v}_1\|^2 = \langle \boldsymbol{v}_1, \boldsymbol{v}_1\rangle = 1 + 9 + 4 = 14$. Hence

$$\boldsymbol{u}_2' = \boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix} - \frac{-31}{14}\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix}.$$

■

2. Find an orthonormal basis of U.

   Sol. Since $\{\boldsymbol{v}_1, \boldsymbol{u}_2'\}$ is an orthogonal basis and $\|\boldsymbol{v}_1\|^2 = 14$,

$$\|\boldsymbol{u}_2'\|^2 = \left\langle \frac{1}{14}\begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix}, \frac{1}{14}\begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix} \right\rangle = \frac{1}{14^2}(9 + 25 + 36) = \frac{5}{14},$$

   $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is an orthonormal basis where

$$\boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \text{ and } \boldsymbol{u}_2 = \frac{\boldsymbol{u}_2'}{\|\boldsymbol{u}_2'\|} = \frac{1}{\sqrt{70}}\begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix}.$$

■

3. Find an orthonormal basis of $\boldsymbol{R}^3$ containing the basis constructed in 2.

   Sol. By Quiz 4-4, we have shown that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{e}_1\}$ is a basis of $\boldsymbol{R}^3$. Hence we can proceed the Gram-Schmidt process one step further to find an orthonomal basis as follows.

$$\boldsymbol{u}_3' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} - \frac{3}{70}\begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \ \boldsymbol{u}_3 = \frac{\boldsymbol{u}_3'}{\|\boldsymbol{u}_3'\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

   Note that $\mathrm{Span}\{\boldsymbol{u}_3\} = U^\perp = \mathcal{N}(A)$. See Quiz 5. ■

4. Find a basis of $U^\perp$.

   Sol. $\dim U^\perp = \dim \boldsymbol{R}^3 - \dim U = 3 - 2 = 1$. Since $\boldsymbol{u}_3 \in U^\perp$ as $U = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ (see Quiz 4), $\{\boldsymbol{u}_3\}$ is a basis of $U^\perp$. Since just a basis is required (not an orhonormal basis), $(2, 0, 1)^T$ is also OK. ■

5. Express each of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ as a linear combination of the orthonormal basis constructed in 3.

   Sol. This is straightforward by a formula in Proposition 7.1.

$$\boldsymbol{e}_1 = \frac{1}{\sqrt{14}}\boldsymbol{u}_1 + \frac{3}{\sqrt{70}}\boldsymbol{u}_2 + \frac{2}{\sqrt{5}}\boldsymbol{u}_3, \boldsymbol{e}_2 = \frac{-3}{\sqrt{14}}\boldsymbol{u}_1 + \frac{5}{\sqrt{70}}\boldsymbol{u}_2, \boldsymbol{e}_3 = \frac{-2}{\sqrt{14}}\boldsymbol{u}_1 - \frac{6}{\sqrt{70}}\boldsymbol{u}_2 + \frac{1}{\sqrt{5}}\boldsymbol{u}_3.$$

■

Take-Home **Quiz 8** (*Due at 7:00 p.m. on Wed. February 14, 2007*)

Division: **ID#:** **Name:**

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and $\boldsymbol{u}$ be vectors in $\boldsymbol{R}^3$ given below.

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -8 \\ -6 \end{bmatrix},\ \boldsymbol{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^3$, let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v}$ be the inner product, $U = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, and $T = \mathrm{proj}_U$. You may quote the facts shown in previous quizzes.

1. Show that $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$, $T(\boldsymbol{v}_2) = \boldsymbol{v}_2$, $T(\boldsymbol{v}_3) = \boldsymbol{v}_3$ and $T(\boldsymbol{u}) = \boldsymbol{0}$.

2. Show that T is a linear transformation using the definition of linear transformations.

3. Show that $T \circ T = T$.

4. Find $\mathrm{Ker}(T)$, $\mathrm{nullity}(T)$, $\mathrm{Im}(T)$ and $\mathrm{rank}(T)$.

5. Show that there is no linear transformation $T' : U \to U$ such that $T'(\boldsymbol{v}_1) = \boldsymbol{v}_2$, $T'(\boldsymbol{v}_2) = \boldsymbol{v}_3$ and $T'(\boldsymbol{v}_3) = \boldsymbol{v}_1$.

Message 欄（何でもどうぞ）：ICU をより魅力的にするにはどうしたらよいでしょうか。また ICU の数学教育について提言があれば。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 8 *(February 14, 2007)*

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and $\boldsymbol{u}$ be vectors in $\boldsymbol{R}^3$ given below.

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -8 \\ -6 \end{bmatrix}, \boldsymbol{u}_1 = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \boldsymbol{u}_2 = \frac{1}{\sqrt{70}}\begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

For $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^3$, let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v}$ be the inner product, $U = \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, and $T = \text{proj}_U$. You may quote the facts shown in previous quizzes.

Recall that $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is an orthonormal basis of U, and $\{\boldsymbol{u}\}$ is a basis of $U^\perp$. Hence

$$\text{proj}_U(\boldsymbol{v}) = \langle \boldsymbol{v}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \langle \boldsymbol{v}, \boldsymbol{u}_2 \rangle \boldsymbol{u}_2. \quad \cdots \quad (*)$$

1. Show that $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$, $T(\boldsymbol{v}_2) = \boldsymbol{v}_2$, $T(\boldsymbol{v}_3) = \boldsymbol{v}_3$ and $T(\boldsymbol{u}) = \boldsymbol{0}$.

   Sol. Using (*), the assertions are easily checked. ■

   Sol. 2. By Proposition 7.1 (b), $\text{proj}_U(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in U$. Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in U$. $\text{proj}_U(\boldsymbol{v}_1) = \boldsymbol{v}_1$, $\text{proj}_U(\boldsymbol{v}_2) = \boldsymbol{v}_2$, $\text{proj}_U(\boldsymbol{v}_3) = \boldsymbol{v}_3$. Since $\boldsymbol{u} \in U^\perp$, $\boldsymbol{u}$ is perpendicular to all basis vectors of U. Hence $\text{proj}_U(\boldsymbol{u}) = \boldsymbol{0}$. ■

2. Show that T is a linear transformation using the definition of linear transformations.

   Sol. Let $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{R}^3$ and k a scalar. Then

$$\begin{aligned} T(\boldsymbol{w}_1 + \boldsymbol{w}_2) &= \text{proj}_U(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \langle \boldsymbol{w}_1 + \boldsymbol{w}_2, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \langle \boldsymbol{w}_1 + \boldsymbol{w}_2, \boldsymbol{u}_2 \rangle \boldsymbol{u}_2 \\ &= \langle \boldsymbol{w}_1, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \langle \boldsymbol{w}_1, \boldsymbol{u}_2 \rangle \boldsymbol{u}_2 + \langle \boldsymbol{w}_2, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \langle \boldsymbol{w}_2, \boldsymbol{u}_2 \rangle \boldsymbol{u}_2 \\ &= \text{proj}_U(\boldsymbol{w}_1) + \text{proj}_U(\boldsymbol{w}_2) = T(\boldsymbol{w}_1) + T(\boldsymbol{w}_2). \\ T(k\boldsymbol{w}_1) &= \langle k\boldsymbol{w}_1, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + \langle k\boldsymbol{w}_1, \boldsymbol{u}_2 \rangle \boldsymbol{u}_2 = k\langle \boldsymbol{w}_1, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1 + k\langle \boldsymbol{w}_1, \boldsymbol{u}_2 \rangle \boldsymbol{u}_2 \\ &= k\text{proj}_U(\boldsymbol{w}_1) = kT(\boldsymbol{w}_1). \quad ■ \end{aligned}$$

   Sol. 2. By Theorem 7.3 (e), every vector $\boldsymbol{v} \in \boldsymbol{R}^3$ is expressed as a sum $\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2$ such that $\boldsymbol{w}_1 \in U$ and $\boldsymbol{w}_2 \in U^\perp$. Clearly $\boldsymbol{w}_1 = \text{proj}_U(\boldsymbol{v})$. Let $\boldsymbol{v}' = \boldsymbol{w}_1' + \boldsymbol{w}_2'$ such that $\boldsymbol{w}_1' \in U$ and $\boldsymbol{w}_2' \in U^\perp$. Then $\boldsymbol{w}_1 + \boldsymbol{w}_1' \in U$ and $\boldsymbol{w}_2 + \boldsymbol{w}_2' \in U^\perp$. Hence $\text{proj}_U(\boldsymbol{v} + \boldsymbol{v}') = \boldsymbol{w}_1 + \boldsymbol{w}_1' = \text{proj}_U(\boldsymbol{v}) + \text{proj}_U(\boldsymbol{v}')$. Similarly $\text{proj}_U(k\boldsymbol{v}) = k\text{proj}_U(\boldsymbol{v})$. ■

3. Show that $T \circ T = T$.

   Sol. Let $\boldsymbol{v} \in \boldsymbol{R}^3$. Since $T(\boldsymbol{v}) = \text{proj}_U(\boldsymbol{v}) \in U$, $T(T(\boldsymbol{v})) = T(\boldsymbol{v})$. Thus $T \circ T = T$. ■

4. Find $\text{Ker}(T)$, $\text{nullity}(T)$, $\text{Im}(T)$ and $\text{rank}(T)$.

   Sol. By definition or the previous problem, $\text{Im}(T) = U$ and $\text{Ker}(T) = U^\perp$. Hence $\text{nullity}(T) = 1$ and $\text{rank}(T) = 2$ as $\dim U^\perp = 1$ and $\dim U = 2$. ■

5. Show that there is no linear transformation $T' : U \to U$ such that $T'(\boldsymbol{v}_1) = \boldsymbol{v}_2$, $T'(\boldsymbol{v}_2) = \boldsymbol{v}_3$ and $T'(\boldsymbol{v}_3) = \boldsymbol{v}_1$.

   Sol. Recall that $\boldsymbol{v}_3 = 5\boldsymbol{v}_1 + \boldsymbol{v}_2$. Hence

$$[1, -3, -2]^T = \boldsymbol{v}_1 = T'(\boldsymbol{v}_3) = T'(5\boldsymbol{v}_1 + \boldsymbol{v}_2) = 5\boldsymbol{v}_2 + \boldsymbol{v}_3 = [-7, 27, 14]^T.$$

   A contradiction. Compare with Proposition 8.3. ■