Review for Final Exam 2015/6

I. True or False. [1]

1. The following is valid for all statements p, q, r.
$$\neg(p \wedge q) \vee r \equiv ((\neg p) \vee r) \wedge ((\neg q) \vee r).$$
2. Let A be an $n \times n$ matrix. If a matrix equation $A\boldsymbol{x} = \boldsymbol{0}$ has infinitely many solutions, so is $A\boldsymbol{x} = \boldsymbol{b}$ for every $n \times 1$ matrix (column vector) $\boldsymbol{b}$.
3. Let A be an $m \times n$ matrix with $m < n$. Then a matrix equation $A\boldsymbol{x} = \boldsymbol{0}$ has infinitely many solutions.
4. Let $f(x)$ be a function such that $f'(c) = 0$ and $f''(c) = 0$. Then $f(x)$ is either increasing, decreasing or a constant at $x = c$, and $f(c)$ cannot be a local extremum.
5. If $F(x)$ is an antiderivative of $f(x)$, then $F(e^x)$ is an antiderivative of $f(e^x)e^x$.

II. Answer each of the following

1. Write a truth table of $(p \Rightarrow q) \vee ((\neg r) \wedge q)$. [2]
2. Find a 3×3 matrix satisfying the following.
$$T \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ -3a + b \\ a + c \end{bmatrix}$$
3. Find BA and AB. [3]
$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 1 & 2 \end{bmatrix}.$$
4. Determine whether or not the matrix A in the previous problem is invertible. [4]
5. Find the reduced row echelon form of the matrix below and find the solutions $x_1, x_2, x_3, x_4, x_5, x_6$. [5]
$$\begin{bmatrix} 1 & 2 & 1 & 0 & -3 & 0 & -2 \\ 0 & 0 & 2 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & -5 \\ 0 & 0 & -3 & 0 & -6 & 3 & 0 \end{bmatrix}$$
6. Find a polynomial $f(x)$ of degree three satisfying $f(1) = 1$, $f(2) = -2$, $f(3) = 4$, $f(4) = 12$. [6]

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[1] I. FFTFT

[2] II-1: same as $p \Rightarrow q$, i.e., TTFFTTTT from top in standard order

[3] II-2: $T = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, II-3: $AB = \begin{bmatrix} 4 & -1 & -3 \\ 4 & 2 & 1 \\ 4 & 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 3 & -1 \\ 2 & 3 & -5 \\ -4 & 5 & 3 \end{bmatrix}$

[4] II-4: $A \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus invertible.

[5] II-5: $\begin{bmatrix} 1 & 2 & 0 & 0 & -5 & 1 & -2 \\ 0 & 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $x_1 = -2s + 5t - u - 2, x_2 = s, x_3 = -2t + u, x_4 = -2t + u - 5, x_5 = t, x_6 = u$.

[6] II-6: $f(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} - 2\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} + 4\frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + 12\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$. II-7. Using $f(x)$ in the previous problem, $f(x) + a(x-1)(x-2)(x-3)(x-4)$, $a \neq 0$ satisfies the condition for any a. Hence there are infinitely many polynomials with the property. II-8. 3.

7. Show that there are infinitely many polynomials $g(x)$ of degree four satisfying $g(1) = 1$, $g(2) = -2$, $g(3) = 4$, $g(4) = 12$.

8. Find the limit. $\lim_{n\to\infty} \dfrac{3n^3 - 4n + 4}{n^3 - 8}$.

9. Find the limit. $\lim_{x\to 2} \dfrac{x^2 - 4x + 4}{x^3 - 8}$.[7]

10. Find the limit. $\lim_{x\to 0} \dfrac{(x+1)e^x - 1}{x}$.

    (Hint: Consider the definition of $f'(0)$ when $f(x) = (x+1)e^x$. There is a straight way. You can apply lHôpital's rule as well.)

11. Find the derivative of $(3x^2 - 2)^{10}$.

12. Find the derivative of $(x^3 - 2x + 1)e^{-3x^2}$.

13. Find $\displaystyle\int \left(x^2 - \frac{2}{x} + \frac{1}{x^2} \right) dx$.

14. Find $\displaystyle\int_1^4 \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx$.

15. Find $\displaystyle\int x(3x^2 - 2)^9 dx$.

16. Find the derivative of $F(x) = \displaystyle\int_0^x t(3t^2 - 2)^9 dt$.

17. Let $y = f(x) = ce^{ax} + de^{bx}$, where a, b, c, d are constants. Suppose $y'' - 2y' - 3y = 0$ and $f(0) = 2$, $f'(0) = 2$. Find a function $y = f(x)$ with these properties by determining constants a, b, c, d. [8]

III. Answer each of the following

1. Find the inverse of the matrix C below, and solve the system of linear equation. [9]

$$C = \begin{bmatrix} 1 & 0 & -1 & -3 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -4 \end{bmatrix}, \quad \begin{cases} x_1 & & -x_3 & -3x_4 & = & 1 \\ x_1 & & -x_3 & -2x_4 & = & 2 \\ & x_2 & & +x_4 & = & -3 \\ & -2x_2 & +x_3 & -4x_4 & = & -1 \end{cases}$$

2. Find a function $f(x)$ satisfying $f'(x) = x^2(x-1)(x-5) = x^4 - 6x^3 + 5x^2$ and $f(0) = 1$. Determine whether $f(x)$ has a local maximum, a local minimum, increasing or decreasing at $x = 0, 1, 5$. [10]

3. Solve the following differential equations.[11]

   (a) $\dfrac{dy}{dx} = 3y,\ y(0) = 3$.

   (b) $\dfrac{dy}{dx} = \dfrac{1}{3\sqrt{xy}},\ y(1) = 1,$ where $x, y > 0$.

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[7] II-9:0, II-10:2, II-11: $60x(3x^2-2)^9$, II-12:$(3x^2-2)e^{-3x^2}+(x^3-2x+1)e^{-3x^2}(-6x) = (-6x^4+15x^2-6x-2)e^{-3x^2}$ II-13:$\frac{1}{3}x^3 - 2\log_e|x| - \frac{1}{x} + C$, II-14: $\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_1^4 = \frac{20}{3}$.

[8] II-15:use II-11 to find $\frac{1}{60}(3x^2-2)^{10} + C$, II-16: $x(3x^2-2)^9$, II-17: $y = e^{3x} + e^{-x}$.

[9] III-1: $C^{-1} = \begin{bmatrix} -4 & 5 & 2 & 1 \\ 1 & -1 & 1 & 0 \\ -2 & 2 & 2 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$, $x_1 = -1, x_2 = -4, x_3 = -5, x_4 = 1$.

[10] III-2: $f(x) = \frac{1}{5}x^5 - \frac{3}{2}x^4 + \frac{5}{3}x^3 + 1$, increasing at 0, local maximum at 1 and local minimum at 5.

[11] III-3: (a) $y = 3e^{3x}$. (b) $y = \sqrt[3]{x}$ or $y^3 = x$.