1 Sets and Logic

Set (集合)： A set is a finite or infinite collection of objects.

The most important requirement when describing a set is that the description makes it clear precisely which elements belong to the set.

「もの」の集まり。どんなものをもってきてもよいが、それがその集まりの中にあるかないかがはっきりと定まっているようなものでなければならない。

Element (元、要素)： The objects that make up the set are called its elements (or members). When a is an element of a set A, we write: $a \in A$ or $A \ni a$, otherwise. $a \notin A$ or $A \not\ni a$

1. $A = \{2, 3, 5, 7\}$
2. $A = \{x \mid x \text{ is a prime at most } 10\} = \{x \mid x$ は 10 以下の素数 $\}$.

Subset (部分集合)： When all elements of a set A belong to a set B, A is said to be a subset of B, and write:

$$A \subset B \text{ or } B \supset A$$

集合 A、B において A のすべての元が、B の元であるとき、A は B の部分集合であると言い $A \subset B$ または、$B \supset A$ と書く。

Equality of Sets (集合の相等)： $A = B$ if and only if $A \subset B$ and $B \subset A$.

二つの集合 A, B において、$A \subset B$ かつ $B \subset A$ が成り立つ時 A と B は相等であると言い $A = B$ と書く。

Empty Set (空集合)： The set with no elements is called the empty set and denoted by $\emptyset$.

元を全く含まない集合を空集合といい $\emptyset$ で表す。

1. $\{y \mid y \text{ is a student in this class of age 100 or above}\} = \emptyset$.
2. $\{x \mid x \text{ is a real number such that } x^2 - 2x + 3 < 0\} = \emptyset$. Note that $x^2 - 2x + 3 = (x-1)^2 + 2 > 0$.

Intersection (共通部分)： The intersection of two sets A and B is the set that contains all elements of A that also belong to B, and is denoted by $A \cap B$.

二つの集合 A, B において、A と B の両方に共通な元全体の集合を A と B との共通部分といい $A \cap B$ と書く。論理記号を用いると

$$A \cap B = \{x \mid (x \in A) \ \wedge \ (x \in B)\} = \{x \mid x \in A, x \in B\}.$$

Union (和集合)： The union of two sets A and B is the set that contains all (distinct) elements in the collection, and is denoted by $A \cup B$.

二つの集合 A, B において、A の元と B の元とを全部寄せ集めて得られる集合を A と B との和集合といい $A \cup B$ と書く。論理記号を用いると

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}.$$

Difference (差集合)： The difference between two sets A and B is the set that contains all elements of A that are not elements of B, and is denoted by $A \setminus B$ or $A - B$.

二つの集合 A, B において、A の元で B の元ではない元全体の集合を A と B との差集合といい、$A \setminus B$ または $A - B$ と書く。論理記号を用いると

$$A \setminus B = \{x \mid (x \in A) \wedge \neg(x \in B)\} = \{x \mid (x \in A) \wedge (x \notin B)\} = \{x \mid (x \in A), (x \notin B)\}.$$

Complement (補集合): A universal set is a set which contains all objects, and U or Ω is often used. When a universal set, say U, is given, the complement of A is $U \setminus A$, and is denoted by A^c or $\overline{A}$.

全体集合 (U または Ω が良く使われる) を一つ定めた時その部分集合 A に対し、A に含まれない要素全体を A^c または $\overline{A}$ で表し、A の補集合と言う。

Statement, Proposition (命題) : A statement is a declarative sentence that is true or false (but not both).

正しいか正しくないが明確に区別できる文を命題という。

Truth Value (真理値) : Every statement has a truth value, namely true (denoted by T (or "1")), or false (denoted by F (or "0").)

命題が真であることを「T (or "1")」、偽であることを「F (or "0")」で表す。これを命題の真理値という。

Negation, logical-and (conjunction), logical-or (disjunction), implication (否定・論理積・論理和・含意):

$\neg p$ (or $\sim p$ or $\bar{p}$ or NOT p), $p \wedge q$ (or $p \cdot q$ or $p\&q$ or p AND q), $p \vee q$ (or $p + q$ or $p \| q$ or p OR q), $p \Rightarrow q$ (or $p \vdash q$ or $p \rightarrow q$ or p IMPLIES q).

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Logically Equivalent (論理同値): Whenever two (compound) statements have the same truth values for all combinations of truth values of their component statements, then we say that X and Y are logically equivalent and write: $X \equiv Y$, otherwise $X \not\equiv Y$.

個々の命題の真理値にかかわらず二つの結合命題 X と Y（命題が $\neg$, $\vee$, $\wedge$, $\Rightarrow$ などで結ばれた式）の真理値が等しいとき、その二つの命題は論理同値（または等値）であるといい $X \equiv Y$ と書く、そうでないとき、$X \not\equiv Y$ と書く。

Propositional Logic and Predicative Logic (命題論理と述語論理)： Propositional logic is the logic the includes sentence letters (p, q, r) and logical connectives $\neg$, $\wedge$, $\vee$, $\Rightarrow$, but not quantifiers.

Predicate logic has the same connectives as propositional logic, but it also has variables for individual objects, quantifiers, such as $\forall x \in A$, $\exists x \in A$.

命題論理は、p, q, r などの、論理命題を $\neg$, $\wedge$, $\vee$, $\Rightarrow$ などで結合させたものをあつかう。

述語命題は、それに加えて、変数を含む論理で、すべての $x \in A$、$x \in A$ が存在するなどの表現をともなう。

Universal Proposition (全称命題) : A proposition asserting something of all things meeting some condition.

In predicate logic, universal quantification formalizes the notion that something (a logical predicate) is true for everything. The resulting statement is a universally quantified statement, and we have universally quantified over the predicate. In symbolic logic, the universal quantifier (typically , $\forall$) is the symbol used to denote universal quantification, and is often informally read as"given any" or "for all".

述語論理において「任意の（すべての）x について命題 $p(x)$ が成り立つ (For all x, the proposition $p(x)$ holds.)」を全称命題といい $\forall x\, p(x)$ または $(\forall x)[p(x)]$ と書く。

Existential Proposition (存在命題) : An existential proposition (or statement) is one affirming the existence of some thing meeting some condition.

In predicate logic, an existential quantification is the predication of a property or relation to at least one member of the domain. It is denoted by the logical operator symbol $\exists$ (pronounced "there exists" or "for some"), which is called the existential quantifier.

述語論理において「ある x について命題 $p(x)$ が成り立つ (There exists x such that $p(x)$ holds.)」を存在命題といい $\exists x\, p(x)$ または $(\exists x)[p(x)]$ と書く。

$$\neg((\forall x)[p(x)]) \equiv (\exists x)[\neg p(x)], \text{ and } \neg((\exists x)[p(x)]) \equiv (\forall x)[\neg p(x)]. \qquad \text{(De Morgan's Laws)}$$

Problems (Find problems of logic in Final Examinations, and Quiz 1 in the World of Mathematics.)

1. Check the following formula by using truth tables. 以下の論理式が論理同値（等値）であることを真理表を用いて確かめよ。
   (a) $p \lor p \equiv p$, $p \land p \equiv p$, $\neg(\neg p) \equiv p$.
   (b) $p \lor q \equiv q \lor p$, $p \land q \equiv q \land p$.
   (c) $(p \lor q) \lor r \equiv p \lor (q \lor r)$, $(p \land q) \land r \equiv p \land (q \land r)$.
   (d) $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$, $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$.
   (e) $\neg(p \land q) \equiv (\neg p) \lor (\neg q)$, $\neg(p \lor q) \equiv (\neg p) \land (\neg q)$. (De Morgan's Laws)
   (f) $p \Rightarrow q \equiv (\neg p) \lor q$.

2. Show the following using 1 (a)-(f) above. 次の論理同値性を 1 (a)-(f) を用いて示せ。
$$p \Rightarrow q \equiv (\neg(\neg q)) \lor (\neg p) \equiv \neg q \Rightarrow \neg p \equiv \neg(p \land (\neg q)).$$

3. Show that the following does not hold by giving an example.
$$(p \land q) \lor r \equiv p \land (q \lor r)$$
上の式が成立しないことを示し、成立しないような命題 p, q, r の例をあげよ。

4. Show $p \Rightarrow (q \lor r) \equiv p \land (\neg q) \Rightarrow r$ and conjugate the following using this formula. $p \Rightarrow (q \lor r) \equiv p \land (\neg q) \Rightarrow r$ であることを示し、それを用いて、次の式を書きかえよ。
$$x(x-1) = 0 \Rightarrow (x = 0) \lor (x = 1).$$

5. AY2005 Mathematical Methods in Science, Quiz 1:
   (a) Let p, q, r be statements. Check the following using the truth table. p, q, r を命題とする。このとき、次の式が成り立つかどうかを、真理表によって判定せよ。理由も記せ。
$$(p \lor q) \Rightarrow r \equiv (p \Rightarrow r) \land (q \Rightarrow r).$$
   (b) Express the following using only $\neg$ and $\lor$. Don't use $\Rightarrow$ and $\land$. $(p \Rightarrow r) \land (q \Rightarrow r)$ を $\neg$ と $\lor$ と 括弧だけを用いて表せ。これらは、何度使っても良いが、$\Rightarrow$ と $\land$ は使わないこと。
   (c) Let X be a compound statement having truth value T exactly when (p,q,r) is (T,F,T), (F,T,F), and (F,F,F). Fill $\neg$, $\land$ or $\lor$ in the underlined blanks below.
$$\begin{aligned} X \quad \equiv \quad & ((((\neg\, p) \,\_\_\_\, (\neg\, q)) \,\_\_\_\, (\neg\, r)) \;\lor \\ & (((\_\_\_\, p) \,\_\_\_\, (\neg\, q)) \;\land\; (\_\_\_\, r))) \,\_\_\_ \\ & (((\_\_\_\, p) \;\land\; (\_\_\_\, q)) \,\_\_\_\, (\_\_\_\, r)) \end{aligned}$$

6. AY2007 Mathematical Methods in Science, Quiz 1:
   (a) Let p, q, r be statements. Check the following using the truth table. Give an explanation by stating the meaning of '$\equiv$'.
$$p \Rightarrow (q \lor r) \equiv (\neg(p \land r)) \Rightarrow q.$$
   (b) For statements p and q, $p \downarrow q$ is a statement having the following truth values.
      i. Complete the truth table of $p \downarrow p$.

p	q	$p \downarrow q$	$p \downarrow p$
T	T	F	
T	F	F	
F	T	F	
F	F	T	

      ii. Write a compound statement logically equivalent to $p \downarrow q$ using only $\neg$ and $\lor$ without using $\land$ or $\Rightarrow$.
      iii. Write a compound statement logically equivalent to $p \lor q$ using only $\downarrow$.

7. AY2007 Mathematical Methods in Science, Final Examination:
Complete the truth tables of $p \Rightarrow (\neg(q \lor r))$ and $\neg((p \land q) \lor (p \land r))$, and check whether these are logically equivalent.

2 System of Linear Equations

Matrices（行列）

Definition 2.1 A *matrix* (or an $m \times n$ *matrix*) is an $m \times n$ rectangular array of numbers. B is a 3×4 matrix, C a 3×3 matrix, and $\boldsymbol{b}$ a 3×1 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \; B = \begin{bmatrix} 3 & -3 & -2 & 16 \\ -3 & 8 & 8 & -25 \\ 1 & -2 & -2 & 7 \end{bmatrix}, \; C = \begin{bmatrix} 3 & -3 & -2 \\ -3 & 8 & 8 \\ 1 & -2 & -2 \end{bmatrix}., \; \boldsymbol{b} = \begin{bmatrix} 16 \\ -25 \\ 7 \end{bmatrix}.$$

上の A の様に $m \times n$ 個の数を長方形（矩形）に並べたを $m \times n$ **行列**、又は、(m, n) 行列と言う。略して、$A = [a_{ij}]$ などと書くこともある。B は 3×4 行列 、C は 3×3 行列、$\boldsymbol{b}$ は 3×1 行列。

Linear Systems and Augmented Matrices（連立一次方程式と拡大係数行列）

Definition 2.2 A finite set of linear equations in variables $x_1, x_2, \ldots, x_n$ is called a *system of linear equations* or a *linear system*, where $x_1, x_2, \ldots, x_n$ are the unknowns. A *solution*（解）of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted（代入する）for $x_1, x_2, \ldots, x_n$, respectively. The set of all solutions of the system is called its *solution set* or the *general solution*（一般解）of the system. Two linear systems are *equivalent*（同値）if they have the same solution set. A system of equations that has no solutions is said to be *inconsistent*（解なし・不能）; if there is at least one solution of the system, it is called *consistent*（解が存在する）. The matrix A above is called the *augmented matrix*（拡大係数行列）of the system, and C the *coefficient matrix.*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \cdots\cdots\cdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{cases},$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & \cdots\cdots\cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}, \quad C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

下の様な $x_1, x_2, \ldots, x_n$ を変数とする 1 次（変数の 1 乗と定数だけを含む）方程式の組を $x_1, x_2, \ldots, x_n$ を未知数とする連立一次（線形）方程式という。 この連立一次方程式において、下の A を**拡大係数行列** とい、C を**係数行列**という。$x_1, x_2, \ldots, x_n$ に代入して等号が成立する数の組 $s_1, s_2, \ldots, s_n$ を**解**といい、その解すべてを表したものを**一般解**、その集合を**解集合**という。解集合が同じ連立一次方程式を**同値**な連立一次方程式という。解が一組でもあるときは、解が存在するといい、一組も無いとき解なしまたは不能という。

Fundamental Questions About a Linear System（連立一次方程式に関する基本的な問題）

1. Is the system consistent? 解は存在するか。
2. If a solution exists, is it the only one? Is the solution unique? 解が存在するとき、それは唯一つか。
3. If more than one solution exist, how can we describe the solution set?
解が一つではないとすると、解はどのように表すことができるか。

Example 2.1

$$\begin{cases} x & - & 3y & = & 2 \\ x & + & 2y & = & 12 \end{cases} \qquad \begin{bmatrix} 1 & -3 & 2 \\ 1 & 2 & 12 \end{bmatrix}, \qquad \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}.$$

$$\begin{cases} x & - & 3y & = & 2 \\ & & 5y & = & 10 \end{cases}, \quad \begin{cases} x & - & 3y & = & 2 \\ & & y & = & 2 \end{cases}, \quad \begin{cases} x & & & = & 8 \\ & & y & = & 2 \end{cases}.$$

Example 2.2

$$\begin{cases} x_1 + 0x_2 + x_3 + 0x_4 + x_5 + 3x_6 & = & -1 \\ -x_1 + 0x_2 - x_3 + 0x_4 + 0x_5 - 4x_6 & = & -1 \\ 0x_1 + x_2 - 2x_3 + 3x_4 + 0x_5 - x_6 & = & 3 \\ -2x_1 - 2x_2 + 2x_3 - 6x_4 - 2x_5 - 4x_6 & = & -4 \end{cases}, \quad \begin{cases} x_1 + 0x_2 + x_3 + 0x_4 + x_5 + 3x_6 & = & -1 \\ -x_1 + 0x_2 - x_3 + 0x_4 + 0x_5 - 4x_6 & = & -1 \\ 0x_1 + x_2 - 2x_3 + 3x_4 + 0x_5 - x_6 & = & 3 \\ -2x_1 - 2x_2 + 2x_3 - 6x_4 - 2x_5 - 4x_6 & = & 4 \end{cases}$$

Elementary operations on equations（方程式の変形）

1. (Replacement) Replace one equation by the sum of itself and a multiple of another.
（何倍かを加える）一つの方程式に、他の方程式の何倍かを加える。
2. (Interchange) Interchange two equations.（交換）方程式の順序を変える。
3. (Scaling) Multiply an equation through by a nonzero constant.（定数倍）方程式を何倍かする。

Elementary row operations（基本行変形）

1. (Replacement) Replace one row by the sum of itself and a multiple of another.（何倍かを加える）ある行に、他の行の何倍かを加える。

   $[i, j; c]$: Replace row i by the sum of row i and c times row j. i 行に j 行の c 倍を加える。
2. (Interchange) Interchange two rows.（交換）二つの行を入れ換える。

   $[i, j]$: Interchange row i and row j. i 行と j 行を入れ換える。
3. (Scaling) Multiply all entries in a row by a nonzero constant.（定数倍）ある行を何倍かする。

   $[i; c]$: Multiply all entries in row i by a nonzero constant c. i 行を c $(c \neq 0)$ 倍する。

Definition 2.3 Two matrices are called *row equivalent* if there is a sequence of elementary row operations that transforms one matrix into the other. 二つの行列が、基本行変形によって移りあうとき、それらは、**行同値**であると言う。

Proposition 2.1 *If an augmented matrix of a linear system is obtained from an augmented matrix of another linear system by a sequence of elementary row operations, i.e., if augmented matrices are row equivalent, then the two systems have the same solution set.*
一つの連立一次方程式の拡大係数行列が、他の連立一次方程式の拡大係数行列に何回か基本行変形を施して得られたとすると（即ち、拡大係数行列が行同値であるなら）、二つの連立一次方程式は同じ解集合をもつ。すなわち同値である。

Definition 2.4 A matrix is said to be in *reduced row-echelon form* if the following conditions hold. 次のような行列を**既約ガウス行列**という。

1. The leftmost nonzero entry (in a nonzero row) is 1. (This is called the *leading* 1.)
ある行が 0 以外の数を含めば、最初の 0 でない数は 1 である。（これを**先頭の** 1 という。）
2. All nonzero rows are above any rows of all zeros.
すべての数が 0 であるような行があれば、その下の行はすべて 0 のみである。
3. Each leading 1 of a row is in a column to the right of the leading 1 of the row above it.
上の行の先頭の 1 は、下の行の先頭の 1 よりも前（左）にある。
4. Each leading 1 is the only nonzero entry in the column.
先頭の 1 を含む列の他の数は、すべて 0 である。

Theorem 2.2 (Gauss-Jordan Elimination) *Every matrix is row equivalent to one and only one matrix in reduced row echelon form.* 任意の行列は、行の基本変形を何回か施して、既約ガウス行列にすることができる。一つの行列と行同値な既約ガウス行列は唯一つである。

Exercise 2.1 Which matrices are in reduced echelon form? Find the reduced echelon form of each of the matrix below. 既約ガウス行列はどれか。既約ガウス行列でないものは、基本変形により既約ガウス行列に変形せよ。

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Definition 2.5 The number of nonzero rows of the reduced row echelon form of a matrix A is called the *rank* of A and write rank A. 行の基本変形で得た既約ガウス行列の 0 でない行の数をその行列の**階数 (rank)** と言い、行列 A に対して、rank A と書く。

Remarks:

1. Since every matrix is row equivalent to one and only one matrix in reduced row echelon form, rank A is determined. Theorem 2.2 によって、どんな行列も基本変形を何回も用いれば既約ガウス行列にすることができ、既約ガウス行列は、唯一つに決まるから、どんな行列にも rank A を決めることができる。

2. If A is in reduced echelon form, rank A is the number of non-zero rows and it is equal to the number of leading 1's. A が最初から既約ガウス行列であれば、rank A は A の 0 でない行の数と等しい。またこの数は「先頭の 1」の数とも等しい。

Theorem 2.3 *Let A be the augmented matrix and C the coefficient matrix of a system of linear equations with n variables. Then the following hold.* n 変数の連立一次方程式の解について以下が成立する。

(1) *If* rank $A \neq$ rank C*, i.e., the reduced echelon form of A has a leading 1 in the last column, the system does not have a solution, i.e., inconsistent.*

拡大係数行列と係数行列の階数が異なれば、すなわち、拡大係数行列の最後の列に先頭の 1 があれば、その連立一次方程式は解を持たない。

(2) *If* rank $A =$ rank $C = n$*, then the system has a unique solution.*

拡大係数行列と係数行列の階数が等しく、その階数が n ならば、その連立一次方程式は丁度一組の解を持つ。

(3) *If* rank $A =$ rank $C < n$*, then there are infinitely many solutions and the general solution can be expressed with $n -$* rank A *free parameters.*

拡大係数行列と係数行列の階数が等しく、階数が $r < n$ ならば、その連立一次方程式の解（の組）は無限個あり、$n - r$ 個の媒介変数を用いて表すことができる。

Remarks: The number of solutions of a system of linear equations is either zero, one or infinity.
連立一次方程式の解（の組）の数は 0 個か、1 個か、無限個かのいずれかである。

Example 2.3 If A is the reduced echelon form of a system of linear equations, then the general solution can be written as follows. 次の行列を拡大係数行列とする方程式の解は次のようになる。

$$A = \begin{bmatrix} 1 & 5 & 0 & 0 & 5 & -1 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1-5t-5u \\ t \\ 1-3u \\ 2-4u \\ u \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u \cdot \begin{bmatrix} -5 \\ 0 \\ -3 \\ -4 \\ 1 \end{bmatrix}$$

Exercise 2.2 [Quiz 2, 2002]

1. By elementary row operations, we obtained the reduced echelon form as follows. 以下のようにある行列に行に関する基本変形を施して、既約ガウス行列を得た。

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & -3 & -6 & 0 & 3 \end{bmatrix} \xrightarrow{(A)} \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -3 & -6 & 0 & 3 \end{bmatrix} \xrightarrow{(B)}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -3 & -6 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix} \xrightarrow{(C)} \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

(1) Write operations in the form $[i, j; c], [i, j]$, or $[i; c]$.
(A), (B), (C) で行なっている行に関する基本変形を $[i, j; c]$, $[i, j]$, $[i; c]$ の何れかの記号で書け。

(A) ☐ (B) ☐ (C) ☐

(2) Find the general solution if the matrix above is the augmented matrix of a system of linear equations in $x_1, x_2, x_3, x_4, x_5, x_6$.
上の行列がある連立一次方程式の拡大係数行列を表す時、その解 $x_1, x_2, x_3, x_4, x_5, x_6$ を求めよ。

Exercises

1. $\left\{\begin{array}{rcrcr} x & - & 3y & = & 2 \\ x & + & 2y & = & 12 \end{array}\right.$ $\qquad \left[\begin{array}{rrr} 1 & -3 & 2 \\ 1 & 2 & 12 \end{array}\right]$

2. $\left\{\begin{array}{rcrcrcr} 3x & - & 3y & - & 2z & = & 16 \\ -3x & + & 8y & + & 8z & = & -25 \\ x & - & 2y & - & 2z & = & 7 \end{array}\right.$

3. $\left\{\begin{array}{rcrcrcr} 3x & - & 3y & + & 3z & = & 15 \\ -3x & + & 8y & - & 8z & = & -25 \\ x & - & 2y & + & 2z & = & 7 \end{array}\right.$

4. $\left\{\begin{array}{rcrcrcr} 3x & - & 3y & + & 3z & = & -2 \\ -3x & + & 8y & - & 8z & = & 8 \\ x & - & 2y & + & 2z & = & -2 \end{array}\right.$

5. $\left[\begin{array}{rrrr} 3 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 \\ 11 & -1 & 5 & 17 \end{array}\right]$

6. $\left[\begin{array}{ccccccc} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ -2 & -2 & 2 & -6 & -2 & -4 & -4 \end{array}\right]$

7. $\left[\begin{array}{ccccccc} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ -2 & -2 & 2 & -6 & -2 & -4 & 4 \end{array}\right]$

3 Matrices

3.1 Matrix Operations: Sum and Scalar Multiple（行列演算：和とスカラー倍）

Definition 3.1 An $m \times n$ *matrix*（行列） (or (m,n) matrix), a matrix of size $m \times n$) is an $m \times n$ rectangular array of numbers with m rows and n columns. An $n \times 1$ matrix is called an n-dimensional *column vector,* and a $1 \times n$ matrix an n-dimensional *row vector.* The vector $\boldsymbol{a}_j$ is the j-th column of the matrix A, and the vector $\boldsymbol{a}'_i$ the i-th row of A. The number $a_{i,j}$ in the i-th row j-th column of a matrix A is called the (i,j) entry, and is also denoted by $A_{i,j}$. Two matrices are defined to be *equal* if they have the same size and their corresponding entries are equal. An $n \times n$ matrix is called a *square matrix*（正方行列）. 下の A の様に $m \times n$ 個の数を長方形（矩形）に並べたを $m \times n$ **行列**、(m,n) 行列、または、型 (m,n) の行列と言う。$1 \times n$ 行列 を n **次元行ベクトル**、$m \times 1$ 行列を m **次元列ベクトル**ともいう。上の行列 A において、左から j 番目の縦に並んだ $\boldsymbol{a}_j$ を A の **第** j **列**と言い、上から i 番目の横に並んだ $\boldsymbol{a}'_i$ を A の**第** i **行**と言う。第 i 行 第 j 列の数を (i,j) **成分**と呼ぶ。下の行列 A は、(i,j) 成分が a_{ij} であるような行列である。(i,j) 成分を $a_{i,j}$ で表す行列という意味で $A = [a_{ij}]$ などと書いたり、行列 A の (i,j) 成分を $A_{i,j}$ と書いたりする。二つの行列は、型が等しく対応する成分がすべて等しいとき、等しいという。$m = n$ であるとき、すなわち $n \times n$ 行列を、n **次正方行列**という。

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ \boldsymbol{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \ \boldsymbol{a}'_i = [a_{i1}, a_{i2}, \ldots, a_{in}].$$

Definition 3.2 Let A and B be matrices of the same size and c a number (called a scalar). Then the *sum*（和） $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A. The *product* cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a *scalar multiple*（スカラー倍） of A. A、B を共に同じ型（$m \times n$）の行列、c を数（スカラー）とする、**和** $A + B$、**スカラー倍** cA を各対応する成分の和と、各成分の c 倍とで定義する。すなわち、

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}, \ cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ & \cdots\cdots\cdots & & \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}.$$

3.2 Products of Matrices（行列の積）

In order to generalize the technique to solve the equation $2x = 6$ or $ax = b$ with $a \neq 0$ to a system of linear equations, we express the following system of linear equations by $A\boldsymbol{x} = \boldsymbol{b}$. $2x = 6$ に、$1/2$ を両辺にかけて、$x = 3$ を得たり、$ax = b$ において、$a \neq 0$ のときに、$x = b/a$ とすることの一般化を連立一次方程式において考える。そのため、下の連立一次方程式を、$A\boldsymbol{x} = \boldsymbol{b}$ と表すことを考える。

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases}, \ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Definition 3.3 Let A be an $m \times n$ matrix above, and $\boldsymbol{x}$ an $n \times 1$ matrix (or n-dimensional column vector). Then the product $A\boldsymbol{x}$ is defined above. 上の $m \times n$ 行列 A と、$n \times 1$ 行列（n-次元ベクトル）$\boldsymbol{x}$ との積を次のように定義する。

$$A\boldsymbol{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots\cdots\cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

Let B be an $\ell \times m$ matrix. Next we want to define BA so that we have $B(A\boldsymbol{x}) = (BA)\boldsymbol{x}$. 次に、$B$ を $\ell \times m$ 行列としたとき、$B(A\boldsymbol{x}) = (BA)\boldsymbol{x}$ が成立するように BA を定義したい。

Example 3.1 Let A, B, $\boldsymbol{x}$ and $\boldsymbol{b}$ be as follows. A, B, $\boldsymbol{x}$, $\boldsymbol{b}$ を以下のようにする。

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix},\ B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix},\ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$$A\boldsymbol{x} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 \\ a_{2,1}x_1 + a_{2,2}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \boldsymbol{b},$$

$$\begin{aligned} B(A\boldsymbol{x}) &= \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}\begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 \\ a_{2,1}x_1 + a_{2,2}x_2 \end{bmatrix} = \begin{bmatrix} b_{1,1}(a_{1,1}x_1 + a_{1,2}x_2) + b_{1,2}(a_{2,1}x_1 + a_{2,2}x_2) \\ b_{2,1}(a_{1,1}x_1 + a_{1,2}x_2) + b_{2,2}(a_{2,1}x_1 + a_{2,2}x_2) \end{bmatrix} \\ &= \begin{bmatrix} (b_{1,1}a_{1,1} + b_{1,2}a_{2,1})x_1 + (b_{1,1}a_{1,2} + b_{1,2}a_{2,2})x_2 \\ (b_{2,1}a_{1,1} + b_{2,2}a_{2,1})x_1 + (b_{2,1}a_{1,2} + b_{2,2}a_{2,2})x_2 \end{bmatrix}, \end{aligned}$$

$$BA = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} b_{1,1}a_{1,1} + b_{1,2}a_{2,1} & b_{1,1}a_{1,2} + b_{1,2}a_{2,2} \\ b_{2,1}a_{1,1} + b_{2,2}a_{2,1} & b_{2,1}a_{1,2} + b_{2,2}a_{2,2} \end{bmatrix}.$$

Definition 3.4 Let $A = [a_{i,j}] = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n]$ be an $m \times n$ matrix with column vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$, and $B = [b_{v,w}]$ an $\ell \times m$ matrix. We define an $\ell \times n$ matrix $C = BA$ by $C = [B\boldsymbol{a}_1, B\boldsymbol{a}_2, \ldots, B\boldsymbol{a}_n]$. Matrix C is called the product of B and A. Each entry of $C = [c_{s,t}]$ is given above. $A = [a_{i,j}] = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n]$ を $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ を A の列ベクトルとする $m \times n$ 行列、$B = [b_{v,w}]$ を $\ell \times m$ 行列とする。このとき、$\ell \times n$ 行列 $C = BA$ を $C = [B\boldsymbol{a}_1, B\boldsymbol{a}_2, \ldots, B\boldsymbol{a}_n]$ で定義する。C を **行列** B **と** A **の積**という。$C = [c_{s,t}]$ の各成分は次のように表される。

$$c_{s,t} = \boldsymbol{b}'_s\boldsymbol{a}_t = [b_{s,1}, b_{s,2}, \ldots, b_{s,m}]\boldsymbol{a}_t = b_{s,1}a_{1,t} + b_{s,2}a_{2,t} + \cdots + b_{s,m}a_{m,t} = \sum_{u=1}^{m} b_{s,u}a_{u,t}.$$

Example 3.2 For a 2×3 matrix A and a 3×2 matrix B we find products AB and BA. 2×3 行列 A と、3×2 行列 B の積 AB と BA を計算する。

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2.3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix} = AB = \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} + a_{1,3}b_{3,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2} \end{bmatrix} = \begin{bmatrix} 10 & 19 \\ 7 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} = BA = \begin{bmatrix} b_{1,1}a_{1,1} + b_{1,2}a_{2,1} & b_{1,1}a_{1,2} + b_{1,2}a_{2,2} & b_{1,1}a_{1,3} + b_{1,2}a_{2,3} \\ b_{2,1}a_{1,1} + b_{2,2}a_{2,1} & b_{2,1}a_{1,2} + b_{2,2}a_{2,2} & b_{2,1}a_{1,3} + b_{2,2}a_{2,3} \\ b_{3,1}a_{1,1} + b_{3,2}a_{2,1} & b_{3,1}a_{1,2} + b_{3,2}a_{2,2} & b_{3,1}a_{1,3} + b_{3,2}a_{2,3} \end{bmatrix} = \begin{bmatrix} 2 & 5 & 9 \\ 3 & 6 & 12 \\ 4 & 7 & 15 \end{bmatrix}.$$

Remarks.

1. For an $m \times r$ matrix A and an $s \times n$ matrix B, the product AB is defined only when $r = s$, in which case AB is an $m \times n$ matrix. $m \times r$ 行列 A と $s \times n$ 行列 B が与えられた時、積 AB がいつも定義できるわけではない。$r = s$ すなわち最初の行列 A の列の数と、後の行列 B の行の数が一致したときに限る。

2. The zero matrix of size $m \times n$ is an $m \times n$ matrix whose entries are all zero. It is denoted by $\mathbf{0} = \mathbf{0}_{m,n}$. すべて成分が零の $m \times n$ 行列を 零行列と言い、$\mathbf{0} = \mathbf{0}_{m,n}$ と書く。

3. The identity matrix $I = I_n$ of size n is a square matrix such that all diagonal entries are 1 and all other entries are 0. If A is an $m \times n$ matrix and B an $n \times m$ matrix, then $AI = A$ and $IB = B$. i 行 i 列の成分（(i,i) 成分）を対角成分と言う。n 次正方行列で、対角成分がすべて 1 他は、すべて 0 であるような行列を、単位行列と言い、$I = I_n$ と書く。(教科書によっては、$E = E_n$ を使っているものも多い。簡単に確かめられるように、A を $m \times n$ 行列、B を $n \times m$ 行列とすると、$AI = A$、$IB = B$。

Proposition 3.1 *The following hold.* 行列の演算に関して次の諸性質が成り立つ。

(1) $A + B = B + A$ (Commutativity (可換律) of Addition)

(2) $A + (B + C) = (A + B) + C$ (Associativity (結合律) of Addition)

(3) $A(BC) = (AB)C$ (Associativity (結合律) of Multiplication)

(4) $A(B + C) = AB + AC$、$(A + B)C = AC + BC$ (Distributivity (分配律))

(5) $cA = (cI)A$

Definition 3.5 An $n \times n$ matrix A is is said to be *invertible*（可逆）(or *nonsingular*（正則）), if there is an $n \times n$ matrix B such that $BA = AB = I$. In this case B is called the *inverse*（逆行列）of A. If no such matrix B can be found, then A is said to be *singular*（非正則）. 正方行列 A について、$BA = AB = I$ を満たす正方行列 B が存在するとき、A は、**可逆**である（又は、可逆行列 (invertible matrix) [**正則行列** (nonsingular matrix)] である）と言う。B を A の逆行列と言い $B = A^{-1}$ と書く。

Example 3.3 Let A be the following matrix. A を下のような 2×2 行列とすると、以下のことがわかる。

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \text{ Then } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix A is invertible if and only if $ad - bc \neq 0$ and A^{-1} is given as follows. A が可逆ということと、$ad - bc \neq 0$ は同値である。

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem 3.2 *Let A be an $n \times n$ square matrix, and $I = I_n$ the identity matrix of size n. Set $C = [A, I]$. If the reduced row echelon form of C is of form $[I, B]$, then $B = A^{-1}$, otherwise the inverse of A does not exist. Thus a square matrix A is invertible if and only if the reduced echelon form of A is I.* A を n 次正方行列、$I = I_n$ を n 次単位行列とし、$C = [A, I]$ なる、$n \times 2n$ の行列を考える。この行列 C に、行に関する基本変形を施し、既約ガウス行列に変形する。その結果を D とする。もし、$D = [I, B]$ の形になれば、$B = A^{-1}$ である。もし、D の左半分が、I で無ければ、A は、逆行列を持たない。とくに、A が逆行列を持つことと、rank $A = n$ であることとは、同値である。

Example 3.4

$$\text{For } A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}, \text{ let } C = C_1 = [A, I] = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

By Jordan-Gaussian elimination, find the reduced row echelon form of C. とおき、行の基本変形を施す。

$$E_1C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} = C_2$$

$$E_2C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -5 & 0 & -3 & 1 \end{bmatrix} = C_3 \; (= E_2E_1C_2)$$

$$E_3C_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -5 & 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -5 & 0 & -3 & 1 \end{bmatrix} = C_4$$

$$E_4C_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -5 & 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -3 & 1 \end{bmatrix} = C_5$$

$$E_5C_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 3 & -1 \end{bmatrix} = C_6$$

$$E_6C_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 5 & -3 & 1 \\ 0 & 0 & 1 & -4 & 3 & -1 \end{bmatrix} = [I, B] = D$$

Hence $D = [I, B] = E_6C_6 = E_6E_5C_5 = E_6E_5E_4E_3E_2E_1C_1 = [E_6E_5E_4E_3E_2E_1A, E_6E_5E_4E_3E_2E_1]$, as $C_1 = [A, I]$. We obtain $I = E_6E_5E_4E_3E_2E_1A$, and $B = E_6E_5E_4E_3E_2E_1$. Thus $I = BA$.

The matrices $E_1, E_2, \ldots, E_6$ corresponding to elementary row operations are called elementary matrices. These are invertible and there are elementary matrices ${E_1}^{-1}, {E_2}^{-1}, \ldots, {E_6}^{-1}$, and $B^{-1} = {E_1}^{-1}{E_2}^{-1}{E_3}^{-1}{E_4}^{-1}{E_5}^{-1}{E_6}^{-1}$. Now $A = B^{-1}BA = B^{-1} = {E_1}^{-1}{E_2}^{-1}{E_3}^{-1}{E_4}^{-1}{E_5}^{-1}{E_6}^{-1}$. Therefore, $AB = I = BA$, $B = A^{-1}$, and A can be written as a product of elementary matrices.

$$A^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ 5 & -3 & 1 \\ -4 & 3 & -1 \end{bmatrix}.$$

Note that, in general, for square matrices A, B of same size, if both A and B are invertible, then $ABB^{-1}A^{-1} = I = B^{-1}A^{-1}AB$. Hence AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Definition 3.6 An $n \times n$ matrix is called an *elementary matrix*（基本行列） if it can be obtained from the $n \times n$ identity matrix I_n by performing a single elementary row operation. 単位行列 $I = I_n$ から一回の基本変形で得られる行列を**基本行列**という。

1. $E(i, j; c)$: the matrix obtained from I_n by performing $[i, j; c]$. $[i, j; c]$ によって得られる行列。
2. $E(i; c)$: the matrix obtained from I_n by performing $[i; c]$ $(c \neq 0)$. $[i; c]$ によって得られる行列。
3. $E(i, j)$: the matrix obtained from I_n by performing $[i, j]$. $[i, j]$ によって得られる行列。

Proposition 3.3 *Let $E(i, j; c)$, $E(i; c)$ and $E(i, j)$ be elementary matrices of size n, and A an $n \times m$ matrix. Then*

$$A \xrightarrow{[i,j;c]} E(i, j; c)A, \quad A \xrightarrow{[i;c]} E(i; c)A, \text{ and } A \xrightarrow{[i,j]} E(i, j)A.$$

Moreover, $E(i, j; c)$, $E(i; c)$ and $E(i, j)$ are invertible and $E(i, j; c)^{-1} = E(i, j; -c)$, $E(i; c)^{-1} = E(i; 1/c)$ and $E(i, j)^{-1} = E(i, j)$.

Proposition 3.4 *For a square matrix A of size n, the following are (logically) equivalent.* A を n 次正方行列とする。次は（論理的に）同値である。

(1) *A is invertible, i.e., there is a square matrix B of size n such that $BA = AB = I$.* A は可逆。すなわち、$BA = AB = I$ を満たす n 次正方行列 B が存在する。

(2) *For each $\boldsymbol{b}$, $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.* $A\boldsymbol{x} = \boldsymbol{b}$ は、$\boldsymbol{b}$ を一つ決めるといつもただ一つの解を持つ。

(3) *$A\boldsymbol{x} = \boldsymbol{0}$ has a unique solution.* $A\boldsymbol{x} = \boldsymbol{0}$ はただ一つの解を持つ。

(4) *I is the reduced row echelon form of A.* A に行の基本変形を施し得られる既約ガウス行列は単位行列 I である。

(5) *A can be written as a product of elementary matrices.* A は、基本行列のいくつかの積で書くことが出来る。

Exercise 3.1 [Quiz 3, 2002]

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \\ -1 & 2 & -5 \end{bmatrix}, \; B = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \; C = [B, I] = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

1. (a) Compute AB. 上の行列の積 AB を計算せよ。

   (b) A is not invertible. Why? 行列 A は逆行列を持たない。理由を述べよ。

2. We want to find the inverse of B as follows. 以下の様にして行列 B の逆行列を求める。

$$C \to C_1 = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to C_2 = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \cdots$$

   (a) C_2 is obtained by multiplying a matrix T to C_1 from the left. Find T and its inverse. C_1 にある行列 T を左からかけると C_2 が得られる。T とその逆行列 S を求めよ。(T、S はそれぞれ $TC_1 = C_2$、$ST = TS = I$ を満たすもの。)

   (b) Find the inverse of B. 行列 B の逆行列を求めよ。

Exercises.

1. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix} =$

2. $\begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} =$

3. 2001, Quiz 3. 1 (a) $\begin{bmatrix} 3 & 8 & 2 \\ 1 & 1 & -1 \\ 4 & 6 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} =$

4. 2002, Quiz 3. 1 (a) $\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \\ -1 & 2 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} =$

5. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ $\qquad$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ $\qquad$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$

6. 2003, Quiz 3. 2 (b) Find A^{-1}.

$$[A, I] = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 7 & 0 & 1 & 0 \\ 2 & 5 & 6 & 0 & 0 & 1 \end{bmatrix} \to$$

7. 2004, Quiz 3. 2 (b) Find A^{-1}.

$$[A, I] = \begin{bmatrix} 1 & -3 & 6 & 1 & 0 & 0 \\ -1 & 3 & -5 & 0 & 1 & 0 \\ 2 & -5 & 8 & 0 & 0 & 1 \end{bmatrix} \to$$

8. 2005, Quiz 3. 3. Find A^{-1}.

$$[A, I] = \begin{bmatrix} 1 & 0 & -3 & 1 & 0 & 0 \\ -1 & 4 & -22 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \to$$

4 Polynomials and Functions

4.1 Polynomials（多項式）

Definition 4.1 Let $c_0, c_1, \ldots, c_n$ be numbers. The expression below involving a symbol x is called a *polynomial* in x. If $c_n \neq 0$, $f(x)$ is of *degree* n and denoted $\deg f(x) = n$. When we consider the correspondence between a number x and the value of $f(x)$, $f(x)$ is called a *polynomial function.* We define $\deg 0 = -\infty$.

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0.$$

$c_0, c_1, \ldots, c_n$ を数とする時、上のような文字 x を含む式を（x に関する）多項式という。$c_n \neq 0$ のとき、$f(x)$ を 次数 n の多項式といい、$\deg f(x) = n$ と書く。x に数を代入して、$f(x)$ の値との対応を考える場合は、$f(x)$ を多項式関数という。$\deg 0 = -\infty$ と約束する。

Theorem 4.1 *Let $f(x)$ and $g(x)$ be polynomials. Then the following hold.* $f(x)$, $g(x)$ を多項式とする。

(1) $\deg f(x)g(x) = \deg f(x) + \deg g(x)$.

(2) *If $g(x) \neq 0$, then there are unique polynomials $q(x)$, $r(x)$ satisfying the following conditions.*

(4–1) $$f(x) = q(x)g(x) + r(x),\ \deg r(x) < \deg g(x).$$

$g(x) \neq 0$ ならば、多項式 $q(x)$, $r(x)$ で 上を満たすものがただ一組存在する。

(3) *Let $a_1, a_2, \ldots, a_m$ be distinct numbers. If $f(a_1) = f(a_2) = \cdots = f(a_m) = 0$, then there is a polynomial $g(x)$ satisfying the following.*

$$f(x) = (x - a_1)(x - a_2)\cdots(x - a_m)g(x),\ \deg g(x) = \deg f(x) - m.$$

$a_1, a_2, \ldots, a_m$ を相異なる数とする。$f(a_1) = f(a_2) = \cdots = f(a_m) = 0$ ならば多項式 $g(x)$ で上の式をみたすものが存在する。

Exercise 4.1 $f(x) = 2x^4 - x^3 + 2x + 1$, $g(x) = x^2 - 2x + 4$, $h(x) = 2x^2 + 3x - 2$, $r(x) = -14x + 9$.

1. $\deg f(x) =$, $\deg g(x) =$, $\deg h(x) =$, $\deg r(x) =$, $\deg 29 =$, $\deg 0 =$.
2. $\deg g(x)h(x) =$, $g(x)h(x) =$
3. $g(x)h(x) + r(x) =$
4. $f(2) =$. Find a polynomial $q(x)$, and a number s such that $f(x) = q(x)(x - 2) + s$.

Let $a_1, a_2, \ldots, a_m$ be distinct numbers, $P(x) = (x - a_1)(x - a_2)\cdots(x - a_m)$ and $P_i(x) = P(x)/(x - a_i)$. Then for $j \neq i$, $P_i(a_j) = 0$ as $P_i(x)$ has a factor $(x - a_j)$, and $P_i(a_i) \neq 0$. Let $Q_i(x) = P_i(x)/P_i(a_i)$. Then $Q_i(a_j) = 0$ and $Q_i(a_i) = 1$. $a_1, a_2, \ldots, a_m$ を相異なる数とする。$P(x) = (x - a_1)(x - a_2)\cdots(x - a_m)$、$P_i(x) = P(x)/(x - a_i)$ とすると、$j \neq i$ のときは、$P_i(x)$ は $(x - a_j)$ の因子を含むから $P_i(a_j) = 0$ となる。また、$P_i(a_i) \neq 0$ である。そこで $Q_i(x) = P_i(x)/P_i(a_i)$ とすると、$Q_i(a_j) = 0$、$Q_i(a_i) = 1$ となる。

Proposition 4.2 (Lagrangian Interpolation, ラグランジュ補間公式) *Let $a_1, a_2, \ldots, a_m$ be distinct numbers. Then there is a polynomial $f(x)$ satisfying $f(a_1) = b_1$, $f(a_2) = b_2, \ldots, f(a_m) = b_m$. $f(x)$ can be written as follows with a polynomial $h(x)$. Moreover, if $\deg f(x) \leqq m - 1$, such $f(x)$ is uniquely determined.*

(4–2) $$f(x) = b_1 Q_1(x) + b_2 Q_2(x) + \cdots + b_m Q_m(x) + h(x)P(x).$$

$a_1, a_2, \ldots, a_m$ を相異なる数とする。このとき、$f(a_1) = b_1$, $f(a_2) = b_2, \ldots, f(a_m) = b_m$ を満たす多項式 $f(x)$ が存在する。$f(x)$ は、ある多項式 $h(x)$ を用いて、常に上のように書くことができる。特に次数 $\deg f(x) \leqq m - 1$ を満たすものはただひとつだけである。

Example 4.1 The polynomial $f(x)$ in (4–3) is the only polynomial of degree at most two satisfying $f(1) = b_1, f(2) = b_2, f(3) = b_3$. Every polynomial $f(x)$ satisfying $f(1) = b_1, f(2) = b_2, f(3) = b_3$ can be written at the top of the next page, where $h(x)$ is a polynomial.

(4–3) $$\begin{aligned} f(x) &= b_1 \cdot \frac{(x-2)(x-3)}{(1-2)(1-3)} + b_2 \cdot \frac{(x-1)(x-3)}{(2-1)(2-3)} + b_3 \cdot \frac{(x-1)(x-2)}{(3-1)(3-2)} \\ &= \frac{b_1}{2}(x-2)(x-3) - b_2(x-1)(x-3) + \frac{b_3}{2}(x-1)(x-2). \end{aligned}$$

$$\frac{b_1}{2}(x-2)(x-3) - b_2(x-1)(x-3) + \frac{b_3}{2}(x-1)(x-2) + h(x)(x-1)(x-2)(x-3).$$

上の多項式は $f(1) = b_1, f(2) = b_2, f(3) = b_3$ を満たす。また、逆に、このような多項式は、ある多項式 $h(x)$ を用いて上の様に書くことができる。

4.2 Formula （公式）

$a^2 - b^2 = (a-b)(a+b)$, $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, $x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1)$. In general,

$$(4\text{–}4) \quad a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + a^{n-i-1}b^i + \cdots + ab^{n-2} + b^{n-1}) = (a-b)\sum_{i=0}^{n-1} a^{n-1-i}b^i.$$

Exercise 4.2 1. $(a+b)(a-b) =$ $\qquad$ $(a-b)(a^2+ab+b^2) =$ $\qquad$ $x^2 \cdot x^3 =$ $\qquad$ $x \cdot x^{10} =$

2. $x^3 - 1 =$ $\qquad$ $y^3 + 8 = y^3 - (-2)^3 =$

3. $x^4 - y^4 =$ $\qquad$ $y^4 - 1 =$ $\qquad$ $z^4 - 16 =$

4. $a^5 - b^5 =$ $\qquad$ $z^5 - 32 = z^5 - 2^5 =$

4.3 Synthetic Division（組み立て除法）

Let $f(x)$ be a polynomial of degree n, and let $g(x) = x - a$. Then by Theorem 4.1 (2), there are polynomials $q(x)$ and $r(x)$ satisfying

$$f(x) = q(x)(x-a) + r(x), \quad \deg(r(x)) < \deg(g(x)) = \deg(x-a) = 1.$$

Since $\deg(r(x)) < 1$, $r(x)$ is a constant. Let $r(x) = r$. So, $f(x) = q(x)(x-a) + r$. In the following we introduce a method called *synthetic division* to find $q(x)$ and r.

If $f(x)$ is a constant f, then by setting $q(x) = 0$ and $r = f$, we have $f(x) = q(x)(x-a) + r$. So we may assume that $\deg(f(x)) \geq 1$. Then the degree of the left hand side of $f(x) - r = q(x)(x-a)$ is $\deg f(x) = n$. Hence by Theorem 4.1 (1), it is equal to $\deg(q(x)) + \deg(x-a)$

$$n = \deg(f(x)) = \deg(f(x) - r) = \deg(q(x)(x-a)) = \deg(q(x)) + \deg(x-a) = \deg(q(x)) + 1.$$

Thus $\deg(q(x)) = n - 1$. Hence by setting $f(x)$ and $q(x)$ as follows and compare the both hand sides of the equation $f(x) - r = q(x)(x-a)$.

$$\begin{aligned} f(x) &= c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0, \\ q(x) &= b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0. \end{aligned}$$

$$\begin{aligned} f(x) - r &= c_n x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + (c_0 - r), \\ q(x)(x-a) &= (b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0)(x-a) \\ &= b_{n-1}x^n + (b_{n-2} - b_{n-1}a)x^{n-1} + (b_{n-3} - b_{n-2}a)x^{n-2} + \cdots + (b_0 - b_1 a)x + (-b_0 a). \end{aligned}$$

Now comparing the coefficients of $x^n, x^{n-1}, \ldots, x$ and the constant term (the coefficient of x^0), we have

$$c_n = b_{n-1},\ c_{n-1} = b_{n-2} - b_{n-1}a,\ c_{n-2} = b_{n-3} - b_{n-2}a,\ \ldots,\ c_1 = b_0 - b_1 a,\ c_0 - r = -b_0 a.$$

Hence

$$b_{n-1} = c_n,\ b_{n-2} = c_{n-1} + b_{n-1}a,\ b_{n-3} = c_{n-2} + b_{n-2}a,\ \ldots, b_0 = c_1 + b_1 a,\ r = c_0 + b_0 a.$$

Therefore to find $q(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$ and r, first b_{n-1} is c_n, the coefficient of x^n of $f(x)$. b_{n-2} is the sum of c_{n-1} and a times b_{n-1}. b_{n-3} is the sum of c_{n-2} and a times b_{n-2}. Finally, the sum of c_0 and a times $b_0$becomes r. Therefore, we can write as follows.

a	c_n	c_{n-1}	c_{n-2}	$\cdots$	c_1	c_0
		$b_{n-1}a$	$b_{n-2}a$	$\cdots$	$b_1 a$	$b_0 a$
	c_n	$c_{n-1} + b_{n-1}a$	$c_{n-2} + b_{n-2}a$	$\cdots$	$c_1 + b_1 a$	$c_0 + b_0 a\ (= r)$
	$(= b_{n-1})$	$(= b_{n-2})$	$(= b_{n-3})$	$\cdots$	$(= b_0)$	

Example 4.2 Let $f(x) = x^3 - 2x^2 - 5x + 6$ and $g(x) = x - 2$. Find $q(x)$ and r satisfying $f(x) = q(x)(x-2) + r$.

Solution. Since $\deg(f(x)) = 3$, $\deg(q(x)) = 2$. Since $r = f(2)$, we can find r by substituting 2 into x of $f(x)$. Now we use synthetic division to find both $q(x) = b_2x^2 + b_1x + b_0$ and r at a time. Calculation is shown right. By it we have,

2	1	−2	−5	6
		2	0	−10
2	1	0	−5	−4
		2	4	
2	1	2	−1	
		2		
2	1	4		
	1			

$$x^3 - 2x^2 - 5x + 6 = (x^2 - 5)(x - 2) - 4.$$

Similarly, from the third line to the fifth we have that

$$x^2 - 5 = (x + 2)(x - 2) - 1.$$

From the fifth to the seventh we know that

$$x + 2 = (x - 2) + 4$$

by synthetic division. Thus we have

$$\begin{aligned} f(x) &= (((x-2)+4)(x-2)-1)(x-2)-4 = (x-2)^3 + 4(x-2)^2 - (x-2) - 4 \\ &= y^3 + 4y^2 - y - 4 = -4 - y + 4y^2 + y^3, \text{ where } y = x - 2. \end{aligned}$$

In this way, $f(x)$ is written as a polynomial in $x - 2$. This is useful to find valuess of $f(x)$ near 2. For example, if $x = 2.01$, then $(x-2)^3 = 0.01^3 = 0.000001$, $(x-2)^2 = 0.01^2 = 0.0001$ are very small. Hence near 2, the value of $f(x)$ is close to $-4 - (x-2)$. And $-4 - (x-2) + 4(x-2)^2$ gives a better estimate. It is much easier to compute this compared with the computation of the exact value of $f(x)$.

Exercise 4.3 [Quiz 4 (2002) modified] Let $f(x)$ be a polynomial of degree three satisfying $f(1) = 33$, $f(2) = 27$, $f(3) = 14$, $f(4) = 0$. $f(x)$ を 3 次多項式で、$f(1) = 33$, $f(2) = 27$, $f(3) = 14$, $f(4) = 0$ を満たすものとする。

1. Find a polynomial $g(x)$ of degree 2 such that $f(x) = (x-4)g(x)$. 2 次多項式 $g(x)$ で、$f(x) = (x-4)g(x)$ となるものは何か。

2. Find b_1, b_2, b_3, b_4 when $f(x)$ is expressed as follows. $f(x)$ を下のように書く時、b_1, b_2, b_3, b_4 を求めよ。

$$\begin{aligned} f(x) &= b_1(x-2)(x-3)(x-4) + b_2(x-1)(x-3)(x-4) \\ &\quad + b_3(x-1)(x-2)(x-4) + b_4(x-1)(x-2)(x-3). \end{aligned}$$

Exercise 4.4 [Quiz 4 (2003)]

1. Let $f(x)$ be a polynomial of degree 7, and let $g(x)$ be a polynomial of degree 3. Suppose $q(x)$ and $r(x)$ are polynomials satisfying the following. What is the degree of $q(x)$? Why?

$$f(x) = q(x)g(x) + r(x), \ \deg r(x) < 3.$$

$f(x)$ を次数が 7 の多項式、$g(x)$ を次数が 3 の多項式とする。このとき、$q(x)$ と $r(x)$ を多項式で上の式を満たすものとする。 このとき、$q(x)$ の次数はいくつか。その理由も記せ。

2. Find a polynomial $p(x)$ and a number r satisfying $2x^3 - 3x^2 + x + 1 = p(x)(x-2) + r$. $2x^3 - 3x^2 + x + 1 = p(x)(x-2) + r$ となるような多項式 $p(x)$ と数 r を求めよ。

3. Find a_3, a_2, a_1, a_0 satisfying the following. 次を満たす a_3, a_2, a_1, a_0 を求めよ。

$$2x^3 - 3x^2 + x + 1 = a_3(x-2)^3 + a_2(x-2)^2 + a_1(x-2) + a_0.$$

4. Let $h(x)$ be as written below. Find b_0, b_1, b_2, b_3 when $h(0) = 6$, $h(1) = -2$, $h(2) = 10$, $h(3) = -6$.

$h(x) = b_0 \cdot (x-1)(x-2)(x-3) + b_1 \cdot x(x-2)(x-3) + b_2 \cdot x(x-1)(x-3) + b_3 \cdot x(x-1)(x-2)$.

上の $h(x)$ は、$h(0) = 6$, $h(1) = -2$, $h(2) = 10$, $h(3) = -6$ を満たすとする。このとき、b_0, b_1, b_2, b_3 を求めよ。

4.4 Exponential and Logarithmic Functions （指数関数・対数関数）

Definition 4.2 [Exponential Function（指数関数）] For $a > 0$, the function $f(x) = a^x$ is called the exponential function (指数関数) with base a.（a を底とする）. For any real number (実数) x, a^x is defined by (i) - (iii).

(i) $a^0 = 1$, $a^n = \overbrace{a \cdot a \cdots a}^{n \text{ times}}$, $a^{-n} = (1/a)^n$ $(n = 1, 2, 3, \ldots.)$

(ii) $a^{p/q} = \sqrt[q]{a^p}$ for integers (whole number, 整数) p, q with $q > 0$.

(iii) $a^x = \lim_{n\to\infty} a^{b_n}$ (when $b_1, b_2, b_3, \ldots$ converges (収束) to x) [*to be discussed later*]

For example, for $a = 2$, 2^π is defined as the value the sequence 2^3, $2^{3.1}$, $2^{3.14}$, $2^{3.141}, \ldots$ converges to.

Proposition 4.3 (Exponential Law（指数法則）) *Let a, x, y be real numbers such that $a > 0$. $a > 0$ とする。任意の実数 x, y に対して、次が成立する。*

$$a^{x+y} = a^x \cdot a^y, \quad (a^x)^y = a^{xy}.$$

Definition 4.3 [Logarithm（対数）] For $1 \neq a > 0$ we write $x = \log_a b$ when $b = a^x$, and x called the logarithm of b with base a. (a を底とする b の対数) If $b > 0$ there is a unique x satisfying $b = a^x$. $b = a^x \Leftrightarrow x = \log_a b$.

Since $a^0 = 1$, $\log_a 1 = 0$. By definition, $a^{\log_a x} = x$.

Example 4.3 For $a = 10$, $10 = 10^1$, $100 = 10^2$, $100000 = 10^5$, $0.1 = 10^{-1}$, $0.01 = 10^{-2}$. Hence

$$\log_{10} 10 = 1,\ \log_{10} 100 = 2,\ \log_{10} 100000 = 5,\ \log_{10} 0.1 = -1,\ \log_{10} 0.01 = -2.$$

For $a = 2$, $2 = 2^1$, $4 = 2^2$, $1024 = 2^{10}$, $\sqrt{2} = 2^{1/2}$, $1/2 = 2^{-1}$. Hence

$$\log_2 2 = 1,\ \log_2 4 = 2,\ \log_2 1024 = 10,\ \log_2 \sqrt{2} = 0.5,\ \log_2(1/2) = -1.$$

Proposition 4.4 *For $a > 0$, the following hold.*

(i) $\log_a xy = \log_a x + \log_a y$.

(ii) $\log_a x^y = y \log_a x$.

(iii) *If $b > 0$, then* $\log_a x = \dfrac{\log_b x}{\log_b a}$.

Proof. (i) Suppose $b = \log_a x$, $c = \log_a y$. By definition, $x = a^b$, $y = a^c$. Hence $a^{b+c} = a^b \cdot a^c = xy$. Thus $\log_a xy = b + c = \log_a x + \log_a y$.
(ii) Let $b = \log_a x^y$, $c = \log_a x$. Since $a^b = x^y$, $x = a^c$, $x^y = a^b = (a^c)^y = a^{cy}$. Therefore, $\log_a x^y = b = cy = y \log_a x$.
(iii) Let $c = \log_a x$, $d = \log_b a$. Since $x = a^c$, $a = b^d$, $x = a^c = (b^d)^c = b^{cd}$. Thus,

$$(\log_a x)(\log_b a) = cd = \log_b x, \text{ thus } \log_a x = \frac{\log_b x}{\log_b a} \qquad \blacksquare$$

Example 4.4 Suppose $f(x) = c \cdot a^{bx}$. Let $g(x)$ be the logarithm of $f(x)$ of base a. Then

$$g(x) = \log_a(c \cdot a^{mx}) = \log_a c + m \cdot x = m \cdot x + b,\ (b = \log_a c).$$

Therefore, $g(x)$ becomes a polynomial of degree one in x. Thus an exponential function becomes a simple function by taking logarithm.

Example 4.5 Richter Scale (マグニチュード): The Richter magnitude involves measuring the amplitude (height) of the largest recorded wave at a specific distance from the seismic source. Adjustments are included for the variation in the distance between the various seismographs and the epicentre of the earthquakes. The Richter scale is a base-10 logarithmic scale, meaning that each order of magnitude is 10 times more intensive than the last one. In other words, a two is 10 times more intense than a one and a three is 100 times greater. In the case of the Richter scale, the increase is in wave amplitude. That is, the wave amplitude in a level 6 earthquake is 10 times greater than in a level 5 earthquake, and the amplitude increases 100 times between a level 7 earthquake and a level 9 earthquake. The amount of energy released increases 31.7 times [hs: about $32 = 2^5$] between whole number values.

(http://www.sms-tsunami-warning.com/pages/richter-scale)

$$\text{Hiroshima A-bomb} \sim \text{M6} \sim \text{20kton}, \text{ where 1kton} = \text{TNT1kton} \sim 10^9\text{cal} \sim 4.2 \times 10^9 J.$$

Exercises

1. Find a polynomial of degree at most 3 satisfying $f(0) = 4$、$f(1) = 3$、$f(2) = -6$、$f(3) = 1$. [Final 2001, II-4]

2. Find polynomials $Q(x)$ and $f(x)$ of degree at most 3 satisfying $Q(-1) = 1$, $Q(0) = Q(1) = Q(2) = 0$, and $f(-1) = 2$, $f(0) = -1$, $f(1) = 3$, $f(2) = -6$. [Final 2003, II-5]

3. Let $f(x)$ be a polynomial in x satisfying $f(1) = 2,\ f(2) = 7,\ f(3) = 1,\ f(4) = 8$. [Quiz 4-1, 2001]

   (a) Find an $f(x)$ of degree at most three.

   (b) Find an $f(x)$ of degree exactly four.

4. Let $f(x) = 2x^4+9x^3+6x^2-8x+5 = q(x)(x+2)+r = c_4(x+2)^4+c_3(x+2)^3+c_2(x+2)^2+c_1(x+2)+c_0$. Find a polynomial $q(x)$, a constant r, and c_4, c_3, c_2, c_1, c_0. [Final 2004, I-6]

5. Let $Q(x)$ be a polynomial satisfying $Q(-5) = Q(0) = Q(5) = Q(10) = 0$ and $Q(15) = 1$. Find a polynomial $Q(x)$ of degree 4, and a polynomial $Q(x)$ of degree 5. [Final 2004, I-7]

6. Let $f(x)$ be a polynomial in x satisfying $f(1) = -2,\ f(2) = 2,\ f(3) = 14,\ f(4) = 40.$ [Quiz 5-1, 2001]

   (a) Find a polynomial $g(x)$ of degree at most 3 satisfying $g(1) = 1, g(2) = g(3) = g(4) = 0$.

   (b) Find an $f(x)$ of degree at most three.

   (c) Find an $f(x)$ of degree exactly four.

7. Let $f(x) = 2x^4 - 3x^3 - x^2 - 3x + 3 = a_4(x-2)^4 + a_3(x-2)^3 + a_2(x-2)^2 + a_1(x-2) + a_0$. Find a_0, a_1, a_2, a_3, a_4. [Final 2003, III-2(a)]

8. Let $f(x)$ be a polynomial satisfying $f(1) = a_1,\ f(2) = a_2,\ f(3) = a_3,\ f(4) = a_4$. We want to find a polynomial $g(x)$ satisfying $g(1) = a_1,\ g(2) = a_2,\ g(3) = a_3,\ g(4) = a_4,\ g(5) = a_5$ using $f(x)$. [Final 2004, III-B]

   (a) Show that there is a polynomial $h(x)$ such that $g(x) = f(x) + h(x)(x-1)(x-2)(x-3)(x-4)$.

   (b) Show that if $h(x)$ satisfies $h(5) = (a_5 - f(5))/(5-1)(5-2)(5-3)(5-4) = (a_5 - f(5))/24$, then $g(x)$ in (a) always satisfies the condition.

9. Use the values in Example 4.5. [Quiz 5-3, 2005]

   (a) Let x be a positive number. Find a formula to express the energy of an earthquake of Mx, i.e., magnitude x.

   (b) At the earthquake of M9, the energy released is approximately n times the energy released by Hiroshima A-bomb. Find n.

5 Limit of a Sequence and Continuity of a Function

5.1 Limit of a Sequence（数列の極限）

Limit（極限）： When a sequence of numbers $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ approaches to a number α, i.e., if it tends to α, we say α is its *limit* (or $\{a_n\}$ *converges* to α) and write $a_n \to \alpha\,(n \to \infty)$ or $\lim_{n\to\infty} a_n = \alpha$. If there is no limit, it is *divergent*. 数列 $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ が一定の値 α に近づく時、$\{a_n\}$ は α に**収束** (converge) する、または $\{a_n\}$ の**極限値**は α であるといい、記号で $a_n \to \alpha\ (n \to \infty)$, または $\lim_{n\to\infty} a_n = \alpha$ と書く。収束しない数列は**発散**する (diverge) という。(さらに細かく ∞ に発散、$-\infty$ に発散、振動などと区別する場合もあるが、ここでは、収束または発散の二つの区別のみ考えることとする。)

Proposition 5.1 *Suppose* $\lim_{n\to\infty} a_n = \alpha$, $\lim_{n\to\infty} b_n = \beta$. *Then the following hold.*

(1) $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n = c\alpha$, *if c is a constant* (定数).

(2) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \alpha + \beta$.

(3) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = \alpha\beta$.

(4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{\alpha}{\beta}, \quad (b_n \neq 0, \beta \neq 0)$.

Example 5.1 1. $\lim_{n\to\infty} \frac{1}{n} = 0$: converges to 0 (0 に収束).

2. Divergent (以下の場合はすべて発散)

$$\begin{cases} \lim_{n\to\infty} n & = & \infty & : \text{diverges to } +\infty \text{ (正の無限大に発散)}, \\ \lim_{n\to\infty} -n & = & -\infty & : \text{diverges to } -\infty \text{ (負の無限大に発散)}, \\ \lim_{n\to\infty} (-1)^n n & & & : \text{vibration (発散:振動)}. \end{cases}$$

3. If $a_n = r^n$, then the following holds.

$$\lim_{n\to\infty} r^n = \begin{cases} \infty : \text{ diverges to } +\infty \text{ ((正の無限大に) 発散)} & \text{if } r > 1, \\ 1 : \text{ converges to 1 (1 に収束)} & \text{if } r = 1, \\ 0 : \text{ converges to 0 (0 に収束)} & \text{if } |r| < 1 \\ \text{vibration (発散 (振動))} & \text{if } r \leqq -1. \end{cases}$$

Example 5.2 1. $\lim_{n\to\infty} \left(-\frac{2}{7}\right)^n = 0$, $\lim_{n\to\infty} \frac{3^n}{5^n} = \lim_{n\to\infty} \left(\frac{3}{5}\right)^n = 0$. $\lim_{n\to\infty} \left(-\frac{7}{2}\right)^n$ divergent (発散)

2. $\lim_{n\to\infty} 2 - \frac{5}{n} = \lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{5}{n} = 2 - 0 = 2$.

3. $\lim_{n\to\infty} \left(3 - \frac{1}{n}\right)\left(4 + \frac{1}{n}\right) = \left(\lim_{n\to\infty} 3 - \frac{1}{n}\right)\left(\lim_{n\to\infty} 4 + \frac{1}{n}\right) = 3 \cdot 4 = 12$.

4. $\lim_{n\to\infty} \frac{2-n}{3n-5} = \lim_{n\to\infty} \frac{\frac{2}{n} - 1}{3 - \frac{5}{n}} = \frac{\lim_{n\to\infty} \frac{2}{n} - 1}{\lim_{n\to\infty} 3 - \frac{5}{n}} = \frac{-1}{3} = -\frac{1}{3}$.

5. $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^2}} = \frac{0}{1} = 0$.

6. $\lim_{n\to\infty} \frac{n^2+2n+3}{n+2} = \lim_{n\to\infty} \frac{n + 2 + \frac{3}{n}}{1 + \frac{2}{n}} = \frac{\lim_{n\to\infty} n + 2 + \frac{3}{n}}{\lim_{n\to\infty} 1 + \frac{2}{n}} = \lim_{n\to\infty} n + 2 = \infty$. divergent (発散)

7. $\lim_{n\to\infty} \frac{(3n^2+1)(n^3-n+1)}{(n^4+n-2)(2n+3)} = \lim_{n\to\infty} \frac{(3 + (\frac{1}{n})^2)(1 - (\frac{1}{n})^2 + (\frac{1}{n})^3)}{(1 + (\frac{1}{n})^3 - 2(\frac{1}{n})^4)(2 + 3\frac{1}{n})} = \frac{3}{2}$.

8. $\lim_{n\to\infty} \frac{3^n - 2^n}{3^n + 2^n} = \lim_{n\to\infty} \frac{1 - \left(\frac{2}{3}\right)^n}{1 + \left(\frac{2}{3}\right)^n} = 1$.

5.2 Napier's constant （Nepier の数、自然対数の底） e

Let $a_n = (1+\frac{1}{n})^n$. Then $a_1 = 2$, $a_2 = 2.25$, $a_3 = 2.37, \ldots, a_n = 2.59$, $a_{12} = 2.61$, $a_{365} = 2.71\ldots$. It is known that this sequence is increasing and does not exceed 3.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.7182818284590\cdots \tag{5–5}$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1. \tag{5–6}$$

5.3 Limits and Continuity of a Function（関数の極限・連続性）

Definition 5.1 If $f(x)$ approaches to α as x approaches to, but not equals, a, we say that α is the limit of $f(x)$ as x approaches a, and write as follows:

$$f(x)\to\alpha\ (x\to a) \text{ or } \lim_{x\to a} f(x) = \alpha.$$

関数 $f(x)$ において 変数 x が a と異なる値をとりながら a に近づくとき、$f(x)$ が一つの値 α に近づくならば x が a に近づくときの $f(x)$ の極限値は α であるといい、上の様に書く。

Proposition 5.2 *Suppose* $\lim_{x\to a} f(x) = \alpha$, $\lim_{x\to a} g(x) = \beta$. *Then the following hold.*

(1) $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = c\alpha$ *if c is a constant.* (定数)

(2) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = \alpha+\beta$

(3) $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = \alpha\beta$

(4) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{\alpha}{\beta}\quad (g(x)\neq 0,\ \beta\neq 0)$

Definition 5.2 If a function $f(x)$ defined at a satisfies $\lim_{x\to a} f(x) = f(a)$, we say that $f(x)$ is continuous at $x = a$. If a function $f(x)$ is continuous for all x, $f(x)$ is said to be continuous. 一般に関数 $f(x)$ において、$\lim_{x\to a} f(x) = f(a)$ が成り立つ時、関数 $f(x)$ は $x = a$ で連続 (continuous) であるという。また、関数が定義されている各点で $f(x)$ が連続であるとき、$f(x)$ は連続である、または連続関数であるという。

Example 5.3 Polynomial functions and a^x with $a > 0$ are continuous for all x. If $f(x)$ and $g(x)$ are continuous, then $f(x)+g(x)$, $f(x)-g(x)$, $c\cdot f(x)$ and $f(x)g(x)$ are continuous. Moreover, $f(x)/g(x)$ is continuous whenever $g(x)\neq 0$. 多項式、a^x $(a > 0)$ などは、各点で連続である。また、$f(x)$, $g(x)$ が共に連続ならば、c を定数とするとき、$f(x)+g(x)$, $f(x)-g(x)$, $c\cdot f(x)$, $f(x)g(x)$ も連続である。さらに、$g(x)\neq 0$ となる点においては、$f(x)/g(x)$ も連続である。

Example 5.4

1. Let us consider the limit of $f(x) = (x^2+7x)/(x+1)$ as x approaches -2. Since both x^2+7x and $x+1$ are polynomials, they are continuous at $x = -2$. Moreover, the denominator $x+1$ is nonzero near $x = -2$. Hence

$$\lim_{x\to -2}\frac{x^2+7x}{x+1} = \frac{\lim_{x\to -2} x^2+7x}{\lim_{x\to -2} x+1} = \frac{(-2)^2+7(-2)}{(-2)+1} = \frac{-10}{-1} = 10.$$

Since $f(-2) = 10$ in this case, $f(x)$ is continuous at $x = -2$.

2. Let us consider the limit of $g(x) = (x^2-5x+4)/(x-4)$ as x approaches 4. Since the denominator is zero at $x = 4$, $g(x)$ is not defined at $x = 4$. However it is defined if $x\neq 4$. Since the numerator is $x^2-5x+4 = (x-4)(x-1)$, $g(x) = x-1$ when $x\neq 4$. Hence

$$\lim_{x\to 4}\frac{x^2-5x+4}{x-4} = \lim_{x\to 4}\frac{(x-4)(x-1)}{x-4} = \lim_{x\to 4} x-1 = 4-1 = 3.$$

3. $\lim_{x\to 0} \frac{x^2 - x}{x} = \lim_{x\to 0} \frac{x(x-1)}{x} = \lim_{x\to 0} x - 1 = -1.$

4. $\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x + 2 = 2 + 2 = 4.$

5. $\lim_{x\to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x+2)}{x-1} = \lim_{x\to 1} x + 2 = 1 + 2 = 3.$

6. $\lim_{x\to 2} \frac{x^3 - 8}{x^2 - x - 2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+1)} = \lim_{x\to 2} \frac{x^2+2x+4}{x+1} = \frac{2^2 + 2\cdot 2 + 4}{2+1} = 4.$

The set $[a,b] = \{x \mid a \leqq x \leqq b\}$ is called a closed interval (閉区間), and $(a,b) = \{x \mid a < x < b\}$ an open interval (開区間).

Proposition 5.3 *Suppose $f(x)$ is a continuous function defined on a closed interval $[a,b]$ and α is in between $f(a)$ and $f(b)$. Then there is a number c in this interval such that $f(c) = \alpha$.* 閉区間 $[a,b]$ 上で連続な関数 $f(x)$ において、α を、$f(a)$ と、$f(b)$ の間の値とすると、$f(c) = \alpha$ となる点 c が、区間 $[a,b]$ 内にある。

Example 5.5 Let $f(x) = 4x^5 - 10x^4 - 20x^3 + 40x^2 + 16x - 15$. Then $f(-2) = -15$, $f(-1) = 15$, $f(0) = -15$, $f(1) = 15$, $f(2) = -15$, $f(3) = 15$. Hence in each of the five intervals $[-2,-1]$, $[-1,0]$, $[0,1]$, $[1,2]$ and $[2,3]$ has at least one point x with $f(x) = 0$, i.e., zero of $f(x)$. Since the degree of $f(x)$ is five, there are at most five zeros. [1]Thus in each interval above has exactly one zero.

Proposition 5.4 *f $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ takes a maximal and a minimal value on this interval.* 閉区間 $[a,b]$ 上で連続な関数 $f(x)$ は、$[a,b]$ 上の最大・最小をとる。

Exercises

1. Find the limit of the following. 次の極限を求めよ。[Quizzes 4, 5 (2001), Final 2001 II-6, modified]

   (a) $\lim_{n\to\infty} \frac{n^3 + 2}{n^3}$

   (b) $\lim_{n\to\infty} \frac{n - n^2}{n^2 + 2}$

   (c) $\lim_{n\to\infty} \frac{n^2 - 1}{3n^2 - n - 2}$ [Final 2001 II-6]

2. Find the limit of the following. 次の極限を求めよ。 [Quiz 5, (2001), Quiz 5-1 (2002), Final 2002]

   (a) $\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$

   (b) $\lim_{x\to 3} \frac{x^2 - 2x - 3}{x^2 + x - 6}$

   (c) $\lim_{x\to 3} \frac{x^2 + x - 6}{x^2 - 2x - 3}$

   (d) $\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 - 2x - 3}$

   (e) $\lim_{x\to -3} \frac{x^3 + 27}{x^2 + 2x - 3}$

   (f) $\lim_{x\to 2} \frac{2x^4 - 15x^3 + 42x^2 - 52x + 24}{x^5 - 5x^4 + 5x^3 + 10x^2 - 20x + 8}$

   (g) $\lim_{x\to 1} \frac{x^3 + x^2 - 5x + 3}{x^4 - x^3 - x + 1}$ [Final 2002]

[1]By Theorem 4.1, every polynomial of degree n has at most n zeros. n 次多項式の根は高々n 個。

(h) $\lim_{x\to 0} \frac{e^x - 1}{e^x}$ $(e^0 = 1)$ [Final 2002]

(i) $\lim_{x\to -2} \frac{x^2 + 5x + 6}{x^3 + 8}$ [Final 2003, II-6]

3. Let $f(x) = x^3 - 2x^2 - 5x + 3$. Then $f(-1) = 5, f(0) = 3, f(1) = -3, f(2) = -7, f(3) = -3, f(4) = 15$. [Quiz 5-2 (2002)]

   (a) Determine all intervals below including a zero of $f(x) = 0$. 次の区間のうち、$f(x) = 0$ を満たす x を含むものをすべて決定せよ。

   $$[-1, 0] \quad [0, 1] \quad [1, 2] \quad [2, 3] \quad [3, 4]$$

   (b) Does $f(x)$ has a zero in $[-2, -1]$? Why? 区間 $[-2, -1]$ に、$f(x) = 0$ を満たす x を含むか。理由も述べよ。

4. Find the limit of the following. Show work. If it diverges, give a brief explanation. [Quiz 5-1, 2003]

   (a) $\lim_{n\to\infty} \frac{2n - 5}{2 - 5n}$

   (b) $\lim_{n\to\infty} \left(\frac{-5}{7}\right)^n$

   (c) $\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$

   (d) $\lim_{x\to 2} \frac{x^2 - 2x - 3}{x^2 + x - 6}$

   (e) $\lim_{x\to 2} \frac{x^2 + x - 6}{x^2 - 2x - 3}$

   (f) $\lim_{x\to 2} \frac{(x-2)(x^3 - 2x^2 + 5x - 8)}{x^3 - x^2 - 4}$

5. A polynomial $f(x) = x^4 - 6x^3 + 8x^2 + 10x - 19$ satisfies $f(1) = -6$. Is there a zero of $f(x)$ (x with $f(x) = 0$) within the interval $[1, 2]$? Give an explanation of your answer. [Quiz 5-2, 2003]

6. Find the limit of the following. Show work. If it diverges, give a brief explanation. [Quiz 5-1, 2004]

   (a) $\lim_{n\to\infty} \frac{3n + 2}{1 - 7n}$

   (b) $\lim_{n\to\infty} \left(\frac{-7}{8}\right)^n$

   (c) $\lim_{x\to 2} \frac{x^2 + 4}{x + 2}$

   (d) $\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$

   (e) $\lim_{x\to 2} \frac{x^2 + 4}{x - 2}$

   (f) $\lim_{x\to 2} \frac{x^4 - 16}{x^3 - 3x^2 + 4x - 4}$

7. Explain why the following is valid. [Quiz 5-2, 2004]

   $$\lim_{x\to 2} \frac{(x^3 - x^2 + 5)(x - 2)}{(x^2 + x - 3)(x - 2)} = \lim_{x\to 2} \frac{x^3 - x^2 + 5}{x^2 + x - 3}.$$

6 Derivatives

6.1 Differentiation and Derivatives（微分と導関数）

The derivative of a function is used to determine the rate of change. Hence it is possible to determine whether the function is increasing or decreasing at a point, maximal or minimal values, and the shape of the graph of a function. 微分は関数の変化率（それぞれの点でどのくらいの率で増えているか減っているか）を調べる時に用いられる。それによって、ある点で関数が増えているか、減っているかだけでなく、どこで最大や、最小の値をとるか、その関数のグラフの概形、ある値を何回とるかなどについても調べることができる。

Definition 6.1 If a function $f(x)$ is defined at $x = a$ and its neighborhood, and the limit below exists, we say that $f(x)$ is *differentiable at* $x = a$, and the limit is denoted by $f'(a)$, which is called the *derivative of* $f(x)$ *at* a. $f'(a)$ is the slope of the tangent line to the curve $y = f(x)$ at $x = a$. When $f(x)$ has a derivative at each a, the function corresponding a to $f'(a)$ is called the *derivative of* $f(x)$ and denoted by $f'(x)$, df/dx or Df. The process of finding a derivative is called differentiation.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

関数 $f(x)$ が、点 $x = a$ 及びその近くで定義されていて、かつ、上の極限が存在するとき、$f(x)$ は a で微分可能であると言い、この極限値を $f(x)$ の点 a における微分係数と言い、$f'(a)$ と書く。$f'(a)$ は、$x = a$ における曲線 $y = f(x)$ の接線の傾きである。関数 $f(x)$ が、各点 a で微分可能であるとき、a に $f'(a)$ を対応させる関数を $f(x)$ の導関数と言い、$f'(x)$、df/dx、Df で表す。関数 $f(x)$ から、その導関数 $f'(x)$ を求めることを、微分するという。

Thus the derivative $f'(x)$ is defined as follows. この定義から導関数 $f'(x)$ は次のように定義される。

$$\begin{aligned} f'(x) &= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \quad (\text{Set } t - x = h, \text{ or } t = x + h.) \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad (\text{Note that } t \to x \text{ if and only if } h = t - x \to 0.) \end{aligned}$$

When $f(x)$ is differentiable at a, $f(x)$ is continuous at a. Hence $f(x) \to f(a)$ as $x \to a$. 関数 $f(x)$ が点 a で微分可能ならば、点 a で連続である。すなわち x が a に近づくと $f(x)$ の値は $f(a)$ に近づく。

Proposition 6.1 *Let $f(x)$ and $g(x)$ are differentiable functions, and c a constant. Then the following hold.* $f(x)$, $g(x)$ を微分可能な関数、c を定数とすると以下が成り立つ。

(1) $(f(x) + g(x))' = f'(x) + g'(x)$, $(cf(x))' = cf'(x)$.

(2) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

(3) $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$.

Example 6.1 1. (Derivative of a polynomial 多項式の微分) The derivative of $f(x) = x^n$ at $x = a$, and $f'(x)$.

$$\begin{aligned} \lim_{x \to a} \frac{x^n - a^n}{x - a} &= \lim_{x \to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1})}{x - a} \quad (5\text{-}1) \\ &= \lim_{x \to a} (x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1}) \\ &= na^{n-1}. \end{aligned}$$

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$ as above. 従って、$f(x) = x^n$ の導関数は $f'(x) = nx^{n-1}$。

2. (Derivative of an exponential function 指数関数の微分) $(e^x)' = e^x$。We apply the following formula.

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1, \text{ where } e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

$$f'(a) = \lim_{x \to a} \frac{e^x - e^a}{x - a} = \lim_{h \to 0} \frac{e^{a+h} - e^a}{h} = e^a \lim_{h \to 0} \frac{e^h - 1}{h} = e^a.$$

Proposition 6.2 (Chain Rule: Derivative of a composite function 合成関数の微分) *Suppose that $g(x)$ is differentiable at $x = a$ and $f(x)$ is differentiable at $x = b = g(a)$. Set $F(x) = f(g(x))$. Then the derivative of $F(x)$ at $x = a$ is as follows.*

$$F'(a) = \frac{d}{dx}f(g(a)) = f'(g(a))g'(a). \text{ Hence } F'(x) = f'(g(x))g'(x).$$

Proof. Set $F(x) = f(g(x))$. Then

$$\begin{aligned} F'(a) &= \lim_{x\to a}\frac{f(g(x)) - f(g(a))}{x-a} \\ &= \lim_{x\to a}\frac{f(g(x)) - f(g(a))}{g(x)-g(a)}\lim_{x\to a}\frac{g(x)-g(a)}{x-a} \\ &= f'(g(a))g'(a). \end{aligned}$$

Since $g(x)$ is differentiable at $x = a$, it is continuous at a, i.e., $g(x) \to g(a)$ as $x \to a$. ■

Example 6.2 Let $f(x) = x^{1/n}$. Then $f(x)^n = x$. If we set $g(x) = x^n$. $g(f(x)) = x$ and $g'(x) = nx^{n-1}$. Thus,

$$1 = (x)' = (g(f(x)))' = g'(f(x))f'(x) = n(x^{1/n})^{n-1}f'(x). \text{ Therefore, } f'(x) = (x^{1/n})' = \frac{1}{n}x^{\frac{1}{n}-1}.$$

It follows from Proposition 6.1 (3) that $(x^{-n})' = (1/x^n)' = -nx^{-n-1}$. Hence if n is a positive integer (整数) and m an integer,

$$(x^{m/n})' = \frac{m}{n}x^{\frac{m}{n}-1}.$$

Inverse Function 逆関数 If we solve $y = f(x) = 2x-1$ for x, we have $x = \frac{1}{2}(y+1)$. Let $g(y) = \frac{1}{2}(y+1)$. Then $y = f(x) \leftrightarrow x = g(y)$. In this way, if functions $f(s)$, $g(t)$ satisfies $t = f(s) \leftrightarrow s = g(t)$, i.e., $g(f(s)) = s$, $f(g(t)) = t$, we say that $g(t)$ is an inverse function (逆関数) of $f(s)$. Note that $f(s)$ is an inverse function of $g(t)$. We use the same variable and say, for example, the inverse function of $f(x) = 2x - 1$ is $g(x) = \frac{1}{2}(x+1)$.

Proposition 6.3 *Suppose $f(s)$ and $g(x)$ are mutual inverse functions on an interval. Then*

$$f'(g(t))g'(t) = 1, \text{ or } \quad \frac{dg}{dt} = \frac{1}{(\frac{df}{ds})}.$$

Proof. Applying the chain rule to a composite function, $f(g(t)) = t$, we have $f'(g(t))g'(t) = 1$. Here $f'(g(t))$ is a derivative with respect to s. ■

Example 6.3 The inverse function of e^x is $y = \log x$ $(= \log_e x)$. Since $x = e^y$, we have

$$(\log x)' = \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Example 6.4 Find derivatives. 微分せよ。

1. $y = 4x^3 + 5x^2 - 3x$, $y' = 4\cdot(x^3)' + 5\cdot(x^2)' - 3\cdot(x)' = 12x^2 + 10x - 3$.
2. $y = 2x^3 - 5x^2 - 3$, $y' = 6x^2 - 10x$.
3. $y = (3x+1)(x^2+x+2)$, $y' = 3(x^2+x+2) + (3x+1)(2x+1) = 9x^2 + 8x + 7$.
4. $y = (x^2+1)(x^3-x^2)$, $y' = 2x(x^3-x^2) + (x^2+1)(3x^2-2x) = 5x^4 - 4x^3 + 3x^2 - 2x$.
5. $y = \frac{1}{x^3}$, $y' = \frac{-3}{x^4}$.
6. $y = \frac{7x-6}{x^2+1}$, $y' = \frac{-7x^2+12x+7}{(x^2+1)^2}$.
7. $y = \frac{1}{x+3}$, $y' = \frac{-1}{(x+3)^2}$.
8. $y = x^2e^{-x} = \frac{x^2}{e^x}$, $y' = (2x - x^2)e^{-x}$.
9. $y = x\log x - x$, $y' = \log x$..

l'Hôpital's Rule If the limit of the right hand side exists, then the limit of the left hand side exists and the equality holds.

$$\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \quad \text{if } f(a)=g(a)=0 \qquad \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

We have a semilar result when $\lim_{x\to a} f(x) = \pm\infty$ $\lim_{x\to a} g(x) = \pm\infty$.

Example 6.5 Find the limit of the following using the l'Hôpital's Rule. [Final 2003, II-6, 7]

1. $\displaystyle\lim_{x\to -2}\frac{x^2+5x+6}{x^3+8}$

2. $\displaystyle\lim_{x\to 0}\frac{e^{-x}-1+x}{3x^2}$

6.2 Applications of Derivatives（微分の応用）

Definition 6.2 If $f(a) > f(x)$ (resp. $f(a) < f(x)$) when x is near a, we say that $f(a)$ is a *local maximum* (resp. *minimum*) of $f(x)$ at $x = a$. All local maximums and minimums of a function are called *local extrema.* 点 x が、点 a に十分近いときは、常に、$f(a) > f(x)$ が成り立つとき、$f(x)$ は、$x = a$ で、極大になるといい、$f(a)$ をその極大値、点 a を、極大点という。同様にして、点 x が、点 a に十分近いときは、常に、$f(a) < f(x)$ が成り立つとき、$f(x)$ は、$x = a$ で、極小になるといい、$f(a)$ をその極小値、点 a を、極小点という。極大値と極小値を合わせて極値という。

We distinguish between maximal (or minimal) and local maximum (local minimum). The former is the max in an interval while the latter is a local max and there may be many. 最大・最小は、ある区間での最大値・最小値だが、極大・極小は局地的にみた最大・最小である。

Proposition 6.4 (First Derivative Test) *Suppose $f(x)$ is differentiable. Then the following hold.* $f(x)$ が微分可能とする。このとき次が成立する。

(1) *If $f(x)$ has a local etrema at $x = c$, then $f'(c) = 0$.* $x = c$ で極値（極大または極小値）を持てば、$f'(c) = 0$。

(2) *If $f'(c) > 0$, then $f(x)$ is increasing at $x = c$.* $f'(c) > 0$ ならば、$f(x)$ は $x = c$ で増加。

(3) *If $f'(c) < 0$, then $f(x)$ is decreasing at $x = c$.* $f'(c) < 0$ ならば、$f(x)$ は $x = c$ 減少。

(4) *If $f'(c) = 0$, then $f(x)$ is constant.* 常に $f'(x) = 0$ ならば、$f(x)$ は定数関数。

As we can tell whether $f(x)$ is increasing or decreasing by the values of $f'(x)$, we can tell whether $f'(x)$ is increasing or decreasing by the values of $f''(x)$, the derivative of $f'(x)$. Hence we have the following. $f'(x)$ の増加、減少は、$f'(x)$ の導関数 $f''(x)$（$f'(x)$ の導関数) によって分かることを考えれば、次のことが分かります。

Proposition 6.5 (Second Derivative Test) *Suppose $f(x)$ and $f'(x)$ are differentiable. Then the following hold.* $f(x)$ は２回微分可能な関数とする。このとき次が成立する。

(1) *If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum.* $f'(c) = 0$、$f''(c) < 0$ ならば、関数 $f(x)$ は、c で極大値 $f(c)$ を持つ。

(2) *If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local minimum.* $f'(c) = 0$、$f''(c) > 0$ ならば、関数 $f(x)$ は、c で極小値 $f(c)$ を持つ。

Proof. (1) Suppose $f'(c) = 0$ and $f''(c) < 0$. Since $f''(x)$ is the derivative of $f'(x)$, $f''(c) < 0$ if and only if $f'(x)$ is decreasing at $x = c$. It decreases to get $f'(c) = 0$ at $x = c$. Hence if $x < c$, $f'(x) > 0$ and if $x > c$, $f'(x) < 0$ as far as x is near c, Hence when $x < c$, as x approaches to c from left, $f(x)$ is increasing and $f(x)$ deceases after x passes $x = c$. This implies that $f(x)$ takes a local maximum $f(c)$ at c.

(2) Similar. Please write a proof in your words. ■

Example 6.6 [Local Extrema of $f(x) = x^4 - 8x^2 + 10$] Let us consider local extrema (local maximum and local minimum) of $f(x) = x^4 - 8x^2 + 10$. By Proposition 6.4 (1), if $x = c$ is a local extremum, $f'(c) = 0$. So first find the zeros of $f'(x) = 0$. In order to determine whether $f(c)$ with $f'(c) = 0$ is a local maximum or local minimum, or neither.

$$f'(x) = 4x^3 - 16x = 4x(x+2)(x-2),\ f''(x) = 12x^2 - 16.$$

Thus $f'(x) = 0$ if and only if $x = -2, 0, 2$. The values of $f''(x)$ at these points are $f''(-2) = 32 > 0$, $f''(0) = -16 < 0$, $f''(2) = 32 > 0$. Hence by Proposition 6.5, $f(x)$ has alocal minimum $f(-2) = -6$ at $x = -2$, で極小値 、$x = 0$ で極大値 $f(0) = 10$、$x = 2$ で極小値 $f(2) = -6$ をとることがわかります。表に書くと次のようになります。

x		-2			0			2	
$f(x)$	↘	Min. (極小)	↗	↗	Max. (極大)	↘	↘	Min. (極小)	↗
$f'(x)$	$-$	0	$+$	$+$	0	$-$	$-$	0	$+$
		↗			↘			↗	
$f''(x)$		$+$			$-$			$+$	

Exercises

1. True or false. [Quiz 6-1, 2002]

   (a) If $f(x)$ is differentiable at $x = a$ and increasing at $x = 2$, then $f'(a) > 0$.

   (b) If $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exits, then $\lim_{x \to a} f(x) = f(a)$.

2. Find derivatives of the following functions $y = f(x)$. 次の関数 $y = f(x)$ の導関数を求めよ。 [Quiz 6-1, 2001 (modified)]

   (a) $y = x^3 - 3x$

   (b) $y = 3x^4 - 4x^3 - 24x^2 + 48x - 15$

   (c) $y = (x^2 + 3x + 1) \log x$

   (d) $y = x^2 e^{-x}$

   (e) $y = (x^2 - 3x + 1)^8$

   (f) $y = \dfrac{e^x}{x^2 + 1}$

   (g) $(2x + 1)e^{-x^3}$ [Final 2003, II-9]

   (h) $(x^2 + 1)e^{-x^2 - 1}$ [Final 2002, II-10]

   (i) $(2x^3 + 5)^{10}$ [Final 2003, II-8]

3. Find the derivative of $y = f(x)$. [Quiz 6-1, 2002]

   (a) $y = x^3 - x$

   (b) $y = (x + 1)(x^3 - 4x)$

   (c) $y = (x^2 - 2)e^{-x}$

   (d) $y = (3x^2 + 5)^8$

4. Let $f(x) = \dfrac{1}{x^2+1}$. [Quiz 6-3, 2002]

(a) To find the derivative of $f(x)$ at $x = a$ using the definition, complete the following.

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\frac{1}{x^2+1} - \frac{1}{a^2+1}}{x-a} = \lim_{x\to a} \frac{(a^2+1)-(x^2+1)}{(x-a)(x^2+1)(a^2+1)} \\ &= \end{aligned}$$

(b) Determine whether $f(x)$ is increasing or decreasing at $x = 1$.

5. Find the equation of the tangent line to the curve $y = f(x)$ at $x = a$.

(a) $y = f(x) = x^3 - 3x$ at $x = 3$. [Quiz 6-2, 2001 (modified)]

(b) $y = f(x) = x^3 - 2x^2 - 5x + 6$ [E.g. 4.2: $(x-2)^3 + 4(x-2)^2 - (x-2) - 4$]

(c) $y = \dfrac{1}{(3x^2+1)^3}$ at $x = 1$. [Final 2001, II-6]

6. Find the interval where $y = f(x) = x^3 - 3x$ is decreasing. 関数 $y = f(x) = x^3 - 3x$ が減少している x の範囲を求めよ。 [Quiz 6-2, 2001]

7. Let $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$. [Quiz 7-1, 2001]

(a) Find $f''(x)$. $f''(x)$ を求めよ。

(b) Find local extrema, i.e., local maxima and local minima, of $f(x)$. $f(x)$ が極大、極小をとる x の値を求め、その点で極大か極小か判定せよ。

(c) Find the maximal value of $f(x)$, and x when $f(x)$ is maximal in the interval $[-6, 6]$. $-6 \leqq x \leqq 6$ で $f(x)$ の値が一番大きくなるのは x がいくつの時か。

8. Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$. You can use the fact that $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ とするとき以下の問いに答えよ。このとき、$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$ であることは用いて良い。 [Quiz 7-1, 2002]

(a) Find all x satisfying $f'(x) = 0$. $f'(x) = 0$ となる x をすべて求めよ。

(b) Find $f''(x)$. $f''(x)$ を求めよ。

(c) Find local extrema, i.e., determine whether $f(x)$ has a local maximum, local minimum or neither at the values found in (a). (a) でもとめた各 x について、$f(x)$ は極大か、極小か、またはどちらでもないか判定せよ。

(d) Find x with $-3 \leqq x \leqq 3$ such that $f(x)$ takes the minimal value. $-3 \leqq x \leqq 3$ で $f(x)$ の値が一番小さくなるのは x がいくつの時か。

9. Let $f(x) = x^2 e^{-x}$. $f(x) = x^2 e^{-x}$ とする。 [Quiz 7-2, 2001]

(a) Find $f''(x)$. $f''(x)$ を求めよ。

(b) Find local extrema, local maximum and minimum. $f(x)$ が極大、極小をとる x の値を求め、その点で極大か極小か判定せよ。

(c) Draw the graph of $y = f(x)$. $y = f(x)$ のグラフの概形を描け。

10. Suppose $y = f(x)$ satisfies $f'(c) = 0$, $f''(c) = 0$ and $f'''(c) = 1$. Explain that $f(c)$ is not the local maximum or local minimum of $f(x)$. 関数 $y = f(x)$ は $f'(c) = 0$、$f''(c) = 0$、$f'''(c) = 1$ を満たすとする。このとき、$x = c$ で $y = f(x)$ は極大にも極小にもならないことを説明せよ。 [Quiz 7-3, 2001]

11. Suppose $f(x) = a+b(x-1)+c(x-1)^2+d(x-1)^3$ satisfies $f(1) = 1$, $f'(1) = -2$, $f''(1) = 6$, $f'''(1) = -4$. Find a, b, c and d. $f(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3$ が $f(1) = 1$, $f'(1) = -2$, $f''(1) = 6$, $f'''(1) = -4$ を満たす時 a, b, c, d を求めよ。 [Quiz 8-2, 2001]

12. Let $f(x) = 2x^4 - 3x^3 - x^2 - 3x + 3 = a_4(x-2)^4 + a_3(x-2)^3 + a_2(x-2)^2 + a_1(x-2) + a_0$. [Final 2003, III-2]

    (a) Find a_0, a_1, a_2, a_3, a_4.

    (b) Find $f(2)$, $f'(2)$, $f''(2)$, $f'''(2)$, $f''''(2)$.

    (c) Find a polynomial of degree 4 satisfying $g(2) = 1$, $g'(2) = 1$, $g''(2) = 2$, $g'''(2) = 6$, $g''''(2) = 24$.

13. Find x when $f(x) = e^{-x^2}$ is maximal. $f(x) = e^{-x^2}$ が一番大きくなる時の x の値を求めよ。 [Quiz 7-1, 2002]

14. Suppose a function $f(x)$ satisfies $f'(c) = f''(c) = f'''(c) = 0$ and $f''''(c) = 7$. Explain that $f(x)$ has a local minimum at $x = c$. [Final 2002, I-4]

15. Suppose a function $y = f(x)$ satisfies $f'(c) = 0$, $f''(c) = 0$, $f'''(c) = -1$. Determine whether $y = f(x)$ is maximum, minimum or neither maximum nor minimum. 関数 $y = f(x)$ は $f'(c) = 0$、$f''(c) = 0$、$f'''(c) = -1$ を満たすとする。このとき、$x = c$ で $y = f(x)$ 増加しているか、減少しているか、極大になっているか、極小になっているか、どれでもないか。簡単に理由も書いて下さい。 [Quiz 7-2, 2002]

16. Let $f(x) = (x-c)^2 g(x) + d$, where $g(x)$ and $g'(x)$ are differentiable and c and d are constants. Show that if $g(c) > 0$, then $f(x)$ has local minimum d at $x = c$. [Final 2001, III-4]

17. Let $f(x)$ be a function satisfying $f'(x) = x(x+1)^3(x-2) = x^5+x^4-3x^3-5x^2-2x$ and $f(0) = -2$.

    (a) Determine whether $f(x)$ has a local maximum, a local minimum, increasing, or decreasing at $x = -1, 0, 2$.

    (b) Find x when $f(x)$ takes the maximal or the minimal value.

7 Integrals

7.1 Antiderivative and Indefinite Integral 原始関数と不定積分

Definition 7.1 A function $F(x)$ is said to be an *antiderivative* of a function $f(x)$ if the derivative of $F(x)$ is $f(x)$, i.e., $F'(x) = f(x)$. 関数 $F(x)$ の導関数が、$f(x)$ に等しいとき、すなわち、$F'(x) = f(x)$ が成り立つとき、$F(x)$ を、$f(x)$ の原始関数と言う。

When $F(x)$ is an antiderivative, other anti-derivatives can be written as $F(x) + C$, where C is a constant. The collection of all antiderivatives of $f(x)$ is called the *indefinite integral* of $f(x)$ and is denoted as follows. C is called the antiderivative constant. $F(x)$ が $f(x)$ の原始関数のひとつであるとき、ほかの原始関数は $F(x)+C$（C は定数）と書くことができる。そこで、一つには定まらないが、原始関数すべてを表すという意味で、$f(x)$ の不定積分と呼び、次のように書く。C を積分定数と言う。

$$\int f(x)dx = F(x) + C.$$

Example 7.1 1. $\displaystyle\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$ if $\alpha \neq -1$, and $\displaystyle\int \frac{1}{x}dx = \log|x| + C$.

2. $\displaystyle\int e^x dx = e^x + C$.

Exercise 7.1 Find an antiderivative of the following. 原始関数を求めよ。

1. x^5
2. $3 - 2x$
3. $5 + \dfrac{1}{x}$
4. $3e^x$
5. $\dfrac{6}{x^5}$
6. $\sqrt{x}$
7. $(t+3)^2$
8. $(2t+3)^3$
9. e^{-t}

Exercise 7.2 Find the indefinite integral of the following. 不定積分を求めよ。

1. $\displaystyle\int \left(x^3 + \frac{1}{x^4} + \sqrt[3]{x}\right) dx$
2. $\displaystyle\int \left((x+1)^5 + e^{5x} + \frac{5}{x}\right) dx$

7.2 Definite Integral and Fundamental Theorem of Calculus 定積分と、微積分学の基本定理

Definition 7.2 Suppose a function $f(x)$ is a <u>continuous</u> in the interval $[a,b]$. For a partition $\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and a set of real numbers $\{t_1, t_2, \ldots, t_n\}$ with $t_i \in [x_{i-1}, x_i]$, the following sum is called the Riemann sum.

$$R_{\Delta,\{t_i\}}(f) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

関数 $f(x)$ が、区間 $[a,b]$ で連続であるとする。分割 $\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$ と実数 $t_i \in [x_{i-1}, x_i]$ の集合 $\{t_1, t_2, \ldots, t_n\}$ に対して、上は、リーマン和と呼ばれる。

The Riemann sum approaches to a value I, whenever the largest interval in the partition approaches 0, and I is called the definite integral of $f(x)$ on $[a,b]$ and denoted as follows.

$$I = \int_a^b f(x)dx = \lim_{|\Delta|\to 0} \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \text{ where } |\Delta| = \max_{i\in\{1,2,\ldots,n\}} |x_i - x_{i-1}|.$$

リーマン和は、分割 Δ を限りなく細かくしていくとき、一定の値 I に近づく。I を $[a,b]$ 上 $f(x)$ の定積分と言い、上のように書く。

When $a < b$, we make a convention that

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Proposition 7.1 *Suppose functions $f(x)$ and $g(x)$ are continuous on the interval $[a,b]$. Then the following hold.* 関数 $f(x)$ と $g(x)$ は、区間 $[a,b]$ で連続とする。このとき、次が成り立つ。

(1) $\displaystyle\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

(2) $\displaystyle\int_a^b k\cdot f(x)dx = k\int_a^b f(x)dx.$ (k: *constant*)

(3) *Suppose $f(x) \geqq g(x)$ whenever $a \leqq x \leqq b$. Then* $\displaystyle\int_a^b f(x)dx \geqq \int_a^b g(x)dx.$

Proposition 7.2 (Mean Value Theorem for Definite Integrals (積分の平均値の定理)**)** *If a function $f(x)$ is continuous on a closed interval $[a,b]$, then there is a number c ($a < c < b$) such that the following equality holds.*

$$\int_a^b f(x)dx = (b-a)f(c).$$

関数 $f(x)$ が、閉区間 $[a,b]$ 上で連続ならば、ある、$c \in (a,b)$ で、上の条件を満たすものがある。

Theorem 7.3 (Fundamental Theorem of Calculus (微積分学の基本定理)**)** *Suppose a function $f(x)$ is continuous on a closed interval $[a,b]$. Let*

$$G(x) = \int_a^x f(x)dx.$$

Then $G(x)$ is an antiderivative of $f(x)$. Moreover if $F(x)$ is an antiderivative of $f(x)$, the following holds.

$$\int_a^b f(x)dx = F(b) - F(a).$$

関数 $f(x)$ が、閉区間 $[a,b]$ で連続であるとする。$G(x)$ は $f(x)$ の原始関数である。また、$F(x)$ を一つの原始関数とすると、上の式が成り立つ。

Exercise 7.3 Compute the following. 次の定積分の値を求めよ。

1. $\displaystyle\int_0^2 5dt$
2. $\displaystyle\int_0^4 (2t+5)dt$
3. $\displaystyle\int_1^2 \frac{1}{x^2}dx$
4. $\displaystyle\int_{-2}^2 (-4x^3+x+5)dx$

5. $\int_0^2 (x-2)^5 dx$

6. $\int_1^e \frac{2}{u} du$

7. $\int_0^1 e^{-u} du$

Exercise 7.4 Let $f(x) = (x^2 + x + 2)^{99}$. Find the following.

1. $f'(x)$

2. $\int_{-1}^1 (2t+1)(t^2+t+2)^{99} dt$

3. $F'(x)$ when $F(x) = \int_0^x (2t+1)(t^2+t+2)^{99} dt$

7.3 Differential Equations 微分方程式

An equation involving a function $y = f(x)$ together with its derivatives $y' = f'(x)$ (and $f''(x)$, $f'''(x), \ldots$) is called a differential equation. We study a separable type as an example. An objective of considering a differential equation is to find $y = f(x)$ satisfying the equation together with additional conditions. We often write dy/dx for $y' = f'(x)$. 関数 $y = f(x)$ の導関数 $y' = f'(x)$ (または、y'', y''' などの高階導関数) が含まれる方程式を微分方程式 (differential equation) と言う。その中でも基本的でかつ応用例も多い分離型 (separable type) についてのべる。$y = f(x)$ とするとき、y の導関数を dy/dx と書くことがある。微分方程式は、$y = f(x)$ を求めることが目的である。

Example 7.2 Find $y = f(x)$ satisfying the following conditions. $f(0) = c$ is called an initial condition. 次の条件を満たす、$y = f(x)$ を求める。$f(0) = c$ を初期条件と言う。

$$\frac{dy}{dx} = g(x), \quad f(0) = c, \quad \text{e.g. } g(x) = 2x,\ f(0) = 1$$

If $G(x)$ is an antiderivative, $y = f(x) = G(x) + C$, and $c = f(0) = G(0) + C$. Hence $C = c - G(0)$ and $y = f(x) = G(x) + c - G(0)$. When $g(x) = 2x$ and $f(0) = 1$, $y = f(x) = x^2 + 1$. $G(x)$ を $g(x)$ の原始関数の一つとすると、$y = f(x) = G(x) + C$。$c = f(0) = G(0) + C$ より $C = c - G(0)$ となり、$y = f(x) = G(x) + c - G(0)$ を得る。例では $G(x) = x^2$ と取れるから、$y = f(x) = x^2 + 1$.

Proposition 7.4 *Let $H(x)$ be an antiderivative of a function $h(x)$ and $G(y)$ an antiderivative of $g(y)$.* $H(x)$ を 関数 $h(x)$ の原始関数、$G(y)$ を 関数 $g(y)$ の原始関数とする。

$$y' = \frac{dy}{dx} = \frac{h(x)}{g(y)} \Rightarrow G(y) = H(x) + C.$$

Proof. Let $y = f(x)$. Then $G(y) = G(f(x))$. If we differentiate both hand sides using the chain rule, we have

$$(G(f(x)))' = G'(y)f'(x) = g(y)y' = h(x) = H'(x)$$

Therefore, $G(y) = G(f(x)) = H(x) + C$. ■

Suppose we formally write our equation $\frac{dy}{dx} = \frac{h(x)}{g(y)}$ as $g(y)dy = h(x)dx$ by multiplying dx and $g(y)$, and compute $\int g(y)dy = \int h(x)dx$. By our assumption, we have $G(y) = H(x) + C$ as $G(y)$ is an antiderivative of $g(y)$ and $G(x)$ is an antiderivative of $h(x)$. Note that two antiderivatives of a function differ only by a constant. To rationalize this process we need to develop a theory of so called 'differentials' to deal with dx or dy. These equations are called separable type.

この命題は、微分方程式を、形式的に $g(y)dy = h(x)dx$ と変形し、$\int g(y)dy = \int h(x)dx$ と積分した結果が等しいことを主張している。上記の形の微分方程式を分離型という。

Example 7.3 Let y denote the population at time x.

$$(1)\ \frac{dy}{dx} = ky, \qquad (2)\ \frac{dy}{dx} = k(N-y), \qquad (3)\ \frac{dy}{dx} = \frac{k}{N}(N-y)y, \qquad k\text{: constant}$$

If there is no constraint, (1) holds. Here k is the difference between birth rate and death rate. If the space is limited, there is an upper bound of population, which is called carrying capacity (人口扶養力) and denoted by N. When the population is close to this bound, (2) holds. Since there is a lower bound as well, by combining these, (3) holds in general. It is called the logistic model. All of these are separable type differential equations.

$$kx + C = \int k dx = \int \frac{N}{(N-y)y} dy = \int (\frac{1}{y} + \frac{1}{N-y}) dy = \log y - \log(N-y) = \log(\frac{y}{N-y}).$$

Let y_0 be the value of y at $x = 0$. Then $e^C = y_0/(N - y_0)$, and we have the following.

$$e^{kx+C} = \frac{y}{N-y},\ y = \frac{N}{1+e^{-kx-C}} = \frac{N}{1+be^{-kx}} \quad b = \frac{N-y_0}{y_0} = e^{-C}.$$

Exercises

1. Find the following indefinite integrals. [Quiz 8-3, 2001, modified]

   (a) $\int \left(x^2 + 1 + \sqrt{x} - \frac{1}{x^2} \right) dx$

   (b) $\int e^{5x+1} dx$

   (c) $\int (3x+2)^{10} dx$

   (d) $\int x^2(x-1)e^{-3x} dx$ (Hint $y = x^3 e^{-3x}$)

2. Find the following. [Quiz 8-5, 2002]

   (a) $\int (x^7 + 2x^5 - x + 1) dx$

   (b) $\int (e^x + \frac{1}{x^2}) dx$

   (c) $\int (3x+2)^4 dx$

   (d) $\int_0^2 (2x - x^2) dx$

   (e) $\int_1^4 \sqrt{x} dx$

   (f) $\int \frac{6x}{(3x^2+1)^4} dx$ [Final 2001, II-10]

3. Find the following.. [Quiz 8-5, 2003]

   (a) $\int (x^3 + 5x - 3)dx$

   (b) $\int \frac{2}{x^3}dx$

   (c) $\int (\sqrt{x} + e^x)dx$

   (d) $\int_1^2 (3x^2 + 1)dx$

   (e) $\int_1^2 (2x - 3)^5 dx$

4. Find the following. [Quiz 8-1, 2, 3, 2002]

   (a) The derivative of $f(x) = (x^2 + 1)e^{x^2}$.

   (b) $F'(x)$ when $F(x) = \int_0^x (t^2 + 1)e^{t^2} dt$.

   (c) $\int x(x^2 + 2)e^{x^2} dx$.

5. Find the following [Quiz 8-1, 2, 3, 2003]

   (a) The derivative of $f(x) = (x^2 + 1)^{10}$.

   (b) $F'(x)$, when $F(x) = \int_0^x (t^2 + 1)^{10} dt$.

   (c) $\int x(x^2 + 1)^9 dx$

6. Suppose $y = f(x)$ satisfies $y' = f'(x) = 3x^2 - 1$. [Quiz 8-1, 2001]

   (a) Find $f(x)$, when $f(0) = 1$.

   (b) Draw three graphs of $y = f(x)$ satisfying $f(0) = 1$, $f(1) = 2$ and $f(-1) = -1$.

7. Determine the following functions.

(a) $f(x)$ when $f'(x) = 2x^3 - x$ and $f(0) = 1$. [Quiz 8-4, 2002]

(b) $g(x)$ satisfying $g'(x) = x^3(x-2)(x+2) = x^5 - 4x^3$ and $g(0) = 1$. [Final 2001, III-5, modified]

(c) $h(x)$ satisfying $h'(x) = 5x^4 + 2x + 1$, $h(1) = 4$. [Quiz 8-4, 2003]

8. Find the following.

(a) $\int \left(x^3 + 1 + \frac{1}{\sqrt[3]{x}} \right) dx$ [Final 2002, II-11]

(b) $\int \frac{x^2}{(x^3+8)^4} dx$ [Final 2002, II-12]

(c) $\int_0^1 (2x-1)^5 dx$ [Final 2002, II-13]

(d) $F'(x)$, when $F(x) = \int_1^x (t^2+1)e^{-t^2-1} dt$. [Final 2002, II-14]

(e) $\int \left(6x^2 + 1 + \frac{4}{x^5} \right) dx$ [Final 2003, II-10]

(f) $\int x^2(2x^3+5)^9 dx$ [Final 2003, II-11]

(g) $\int_0^1 (3x+2)^4 dx$ [Final 2003, II-12]

(h) $F'(x)$, when $F(x) = \int_{-2}^x (2t+1)e^{-t^3} dt$. [Final 2003, II-13]

(i) $\int \left(5x^4 + 1 + \frac{4}{x^5} - 3\sqrt{x} \right) dx$ [Final 2004, II-12]

(j) $\int \frac{x}{(x^2+5)^6} dx$ [Final 2004, II-13]

(k) $\int_0^1 10(2x+1)^4 dx$ [Final 2004, II-14]

(l) $F'(x)$, when $F(x) = \int_{-2}^x (t^2+5)e^{-t} dt$. [Final 2004, II-15]